EL24 Lab 12 live

Thursday, April 15, 2021 3:59 PM

Ex.
$$\begin{cases} u_t = 3u_{xx}, & 0 < x < \pi,\ t > 0 \\ u_x(0,t) = u_x(\pi,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$(u)_x = (TX)_x = T \cdot X'(x)$

$u_x(0,t) = T(t)\cdot X'(0) = 0$

$u_x(\pi,t) = T(t)\cdot X'(\pi) = 0$

$\boxed{X'(0) = X'(\pi) = 0}$

Suppose $u(x,t) = \boxed{X(x)\,T(t)}$.

we know that $u_t = 3u_{xx}$

$(X(x)T(t))_t = 3(X(x)T(t))_{xx}$

$X(x)\cdot(T(t))_t = 3T(t)\cdot(X(x))_{xx}$

$X \cdot T' = 3T\cdot X''$

$$\boxed{\frac{T'}{3T} = \frac{X''}{X}} = -\lambda \quad \longrightarrow \quad T' = -3T\lambda,\quad X'' = -\lambda X$$

$u_x(0,t) = u_x(L,t) = 0$

$\frac{c_0}{2} + \sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2 t}{L}} \cos\left(\frac{n\pi x}{L}\right)$

$u(0,t) = u(L,t) = 0$

$\sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2 t}{L}} \sin\left(\frac{n\pi x}{L}\right)$

Two eq's:
$$\text{I}\ \begin{cases} T' + 3\lambda T = 0 \\ X'' + \lambda X = 0 \end{cases}\ \text{II}$$

I: $\frac{T'}{T} = -3\lambda \Rightarrow (\ln(T))' = -3\lambda \longrightarrow \frac{T'}{T} = (\ln(T))' = \frac{1}{T}\cdot T'$

$\Rightarrow \ln(T) = -3\lambda t + C$

$\Rightarrow \boxed{T(t) = \tilde{C} e^{-3\lambda t}}$

$\frac{dT}{dt} = -3\lambda T$

$\int \frac{dT}{T} = \int -3\lambda\, dt$

$\ln(T) = -3\lambda t + C$

II 3 cases: $X'' + \lambda X = 0$ If $\lambda < 0 \Rightarrow$ no eigenvalues!

$\lambda = 0$: $X'' = 0 \Rightarrow X' = A \Rightarrow X(x) = Ax + B$

Boundary cond's: $X'(0) = X'(\pi) = 0$.

$X'(x) = A \Rightarrow X'(0) = X'(\pi) = \boxed{A = 0}$

B can be any constant. (Take $B = 1$)

So, $\lambda_0 = 0$ is an eigenvalue with eigenfunction $X_0(x) = 1$

$\lambda > 0$

$\lambda > 0$: $\Rightarrow \boxed{X(x) = c_1\cos(\sqrt{\lambda}x)} + c_2\sin(\sqrt{\lambda}x)$

$X'' + \lambda X = 0 \Rightarrow r = \pm i\sqrt{\lambda}$ $\Rightarrow$ trig sol's

Boundary cond's: $X'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

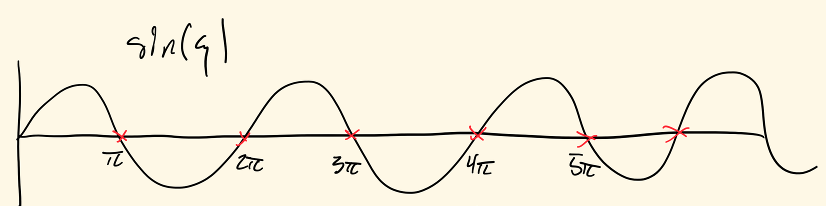
$X'(0) = 0 + c_2\cdot\sqrt{\lambda}\cdot 1 = 0$

$X(x) = c_1\cos(\sqrt{\lambda}x)$

$X'(\pi) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) = 0 \Rightarrow \sin(\sqrt{\lambda}\pi) = 0$

$\Rightarrow \sqrt{\lambda}\,\pi = n\pi$

$\Rightarrow \boxed{\lambda_n = n^2}$



$\sin(q) = 0$ whenever $q = n\pi$

$\boxed{\lambda_n = n^2}$ are the eigenvalues with eigenfunctions:

$\boxed{X_n(x) = C_n\cdot\cos(nx)} \rightarrow$ we can include $n=0$, $\lambda_n = n^2 \Rightarrow \lambda_0 = 0$

$X_0(x) = C_0\cos(\sqrt{\lambda_0}x) = C_0\cdot 1$

Recall

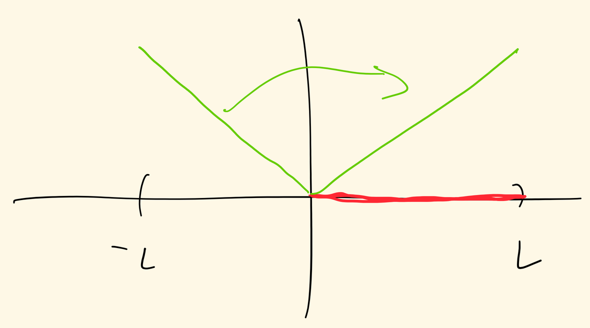
$T(t) = \tilde{C}\cdot e^{-3\lambda t}$

$T_n(t) = \tilde{C}e^{-3\lambda_n t} = \tilde{C}e^{-3n^2t}$ $\boxed{\tilde{C} = 1}$

So,
$$u(x,t) = C_0 + \sum_{n=1}^{\infty} X_n(x)\,T_n(t)$$
$$u(x,t) = C_0 + \sum_{n=1}^{\infty} e^{-3n^2t}\cdot C_n\cos(nx)$$

$\boxed{u(x,0) = f(x)}$

$\sum_{n=0}^{\infty} e^{-3n^2\cdot 0} C_n\cos(nx) = \sum_{n=0}^{\infty} C_n\cos(nx)$



Suppose $f(x) = x$.

$\Rightarrow \sum_{n=0}^{\infty} C_n\cos(nx) = x$

$f(x) = \sum_{n=0}^{\infty} C_n\cos(nx)$

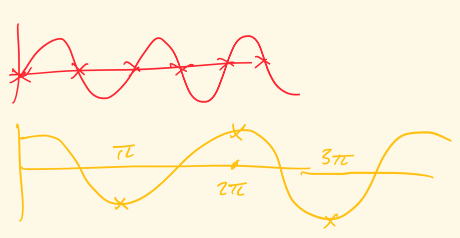
$$C_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\cdot\frac{x^2}{2}\Big|_0^{\pi} = \boxed{\pi}$$

$$C_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

$u = x$, $du = dx$, $dv = \cos(nx)$, $v = \frac{\sin(nx)}{n}$

$$= \frac{2}{\pi}\left[\frac{x\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\sin(nx)}{n}dx\right]$$
$$= \frac{2}{\pi}\left[\frac{x\sin(nx)}{n}\Big|_0^{\pi} - \frac{1}{n}\left(-\frac{\cos(nx)}{n}\Big|_0^{\pi}\right)\right]$$
$$= \frac{2}{\pi}\left[\frac{x\sin(nx)}{n}\Big|_0^{\pi} + \frac{1}{n^2}\cos(nx)\Big|_0^{\pi}\right]$$
$$= \frac{2}{\pi}\left[\frac{\pi\sin(n\pi)}{n} - 0 + \frac{1}{n^2}\left[\underbrace{\cos(n\pi)}_{=(-1)^n} - 1\right]\right]$$
$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$
$$= \begin{cases} 0, & n \text{ odd} \\ -\frac{4}{n^2\pi}, & n \text{ is even} \end{cases} \rightarrow \quad n = \boxed{2K}$$

$u = \boxed{2K-1}$



$u(x,t) = C_0 + \sum_{n=0}^{\infty} e^{-3n^2t} C_n\cos(nx)$

$u(x,0) = C_0 + \sum_{n=1}^{\infty} C_n\cos(nx) = x$

Okay, sol: Find the cosine series for $f(x) = x$ on $(0,\pi)$

$$f(x) = x \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx)$$

$a_0 = 2\cdot\frac{1}{\pi}\int_0^{\pi} x\,dx = \pi$

$a_n = 2\cdot\frac{1}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \begin{cases} 0 \\ -\frac{4}{n^2\pi} \end{cases}$

$\Rightarrow C_0 = \frac{a_0}{2} = \frac{\pi}{2}$

$C_n = a_n = \begin{cases} 0 & \text{even} \\ -\frac{4}{n^2\pi} & \text{odd} \end{cases}$

set $a_n = C_n$

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} e^{-3n^2t} C_n\cos(nx)$$
$$= \pi + \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-3n^2t}\left((-1)^n - 1\right)\cos(nx), \quad \boxed{n = 2K-1}$$
$$\boxed{u(x,t) = \pi - \frac{4}{\pi}\sum_{K=1}^{\infty}\frac{e^{-3(2K-1)^2t}}{(2K-1)^2}\cos((2K-1)x)}$$

Find $\lim_{t\to\infty} u(x,t)$

$1 - e^{-t}\cos(x)$

$u(0,t) = u(L,t) = 0 \rightarrow$ sin series

$u_x(0,t) = u_x(L,t) = 0 \rightarrow$ cos series

$\begin{cases} u(0,t) - u(L,t) = 0 \\ u_x(0,t) - u_x(L,t) = 0 \end{cases} \rightarrow$ Full fourier series.