EL24 Lab 11 live

Thursday, April 8, 2021 3:58 PM

Fourier Series: $f(x)$ on $(-L, L)$, then we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \dots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \dots$$

Even: $f(x) = f(-x)$

Odd: $f(x) = -f(-x)$

even × even = even

even × odd = odd

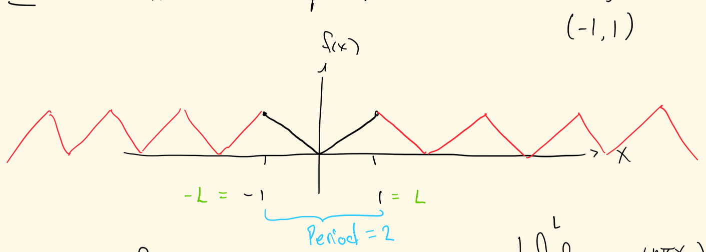
odd · odd = even

$$\int_{-L}^{L}(\text{even})dx = 2\int_0^L(\text{even})dx$$

$$\int_{-L}^{L}(\text{odd})dx = 0$$

Ex. Find the F.S. expansion of $f(x) = |x|$ on $(-1, 1)$

$\boxed{f(x) = |x| = |-x| = f(-x)}$



Since $f(x)$ is even,

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}\,\cdot\,\text{even}\,=\,\text{even}} dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\left(\frac{n\pi x}{L}\right)}_{\text{even}\,\cdot\,\text{odd}\,=\,\text{odd}}dx = 0$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{2}{L}\int_0^L f(x)dx = 2\int_0^1 x\,dx = 2\left.\frac{x^2}{2}\right|_0^1 = \frac{2}{2}\cdot 1 = \boxed{1}$$

$\boxed{L=1}$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 2\int_0^1 x\cdot\cos\left(\frac{n\pi x}{1}\right)dx$$

$u = x$, $du = dx$, $dv = \cos(n\pi x)$, $v = \sin(n\pi x)\frac{1}{n\pi}$

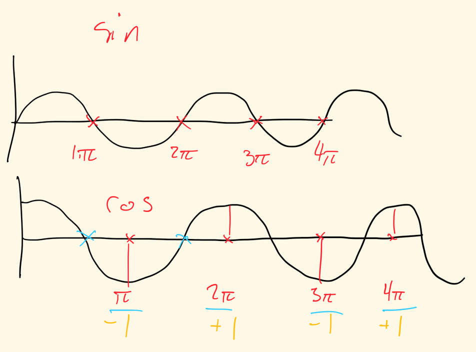
$$= 2\left[\left.\frac{x\sin(n\pi x)}{n\pi}\right|_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi}dx\right]$$

$$= \frac{2}{n\pi}\left[\left. x\sin(n\pi x)\right|_0^1 - \left.\left(-\cos(n\pi x)\cdot\frac{1}{n\pi}\right)\right|_0^1\right]$$

$$= \frac{2}{n\pi}\left[\left. x\sin(n\pi x)\right|_0^1 + \left.\frac{\cos(n\pi x)}{n\pi}\right|_0^1\right]$$

$$= \frac{2}{n\pi}\left[1\cdot\underbrace{\sin(n\pi)}_{=0 \text{ for all } n} - 0 + \underbrace{\frac{\cos(n\pi)}{n\pi}}_{=\frac{(-1)^n}{n\pi}} - \frac{1}{n\pi}\right]$$

$n\pi$ for $n = 1, 2, 3, \dots$



$$= \frac{2}{n\pi}\left[\frac{(-1)^n - 1}{n\pi}\right]$$

$$= \boxed{\frac{2((-1)^n - 1)}{(n\pi)^2}} = \begin{cases} 0, & n \text{ even} \\ -\frac{4}{(n\pi)^2}, & n \text{ odd}\end{cases}$$

So,

$$\boxed{f(x) = |x| = \frac{1}{2} + 2\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{(n\pi)^2}\right)\cos(n\pi x)} \rightsquigarrow \frac{1}{2} - \sum_{K=1}^{\infty}\left(\frac{2}{(2K-1)\pi}\right)^2\cos((2K-1)\pi x)$$

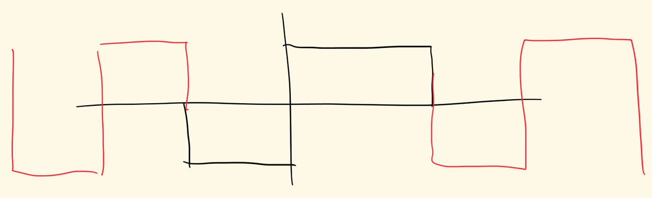
on $(-1, 1)$

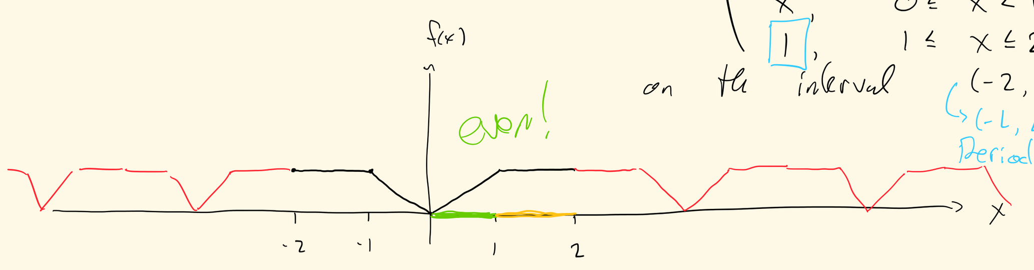
Ex. Find the F.S. expansion of

$$f(x) = \begin{cases} 1, & -2 \le x < -1 \\ -x, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ \boxed{1}, & 1 \le x \le 2 \end{cases}$$

on the interval $(-2, 2)$ $(-L, L)$

Period $= 4 = 2L$





even function on symmetric interval

$\Rightarrow \boxed{b_n = 0}$

$\boxed{L=2}$

$$a_0 = \frac{1}{L}\int_{-L}^{L}f(x) = \frac{2}{L}\int_0^L f(x)dx$$

$$= \frac{2}{2}\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$$

$$= \int_0^1 x\,dx + \int_1^2 1\,dx$$

$$= \left.\frac{x^2}{2}\right|_0^1 + \left. x\right|_1^2 = \frac{1}{2} + 1 = \boxed{\frac{3}{2}}$$

$$|x| = \begin{cases} x & x > 0 \\ -x & x \le 0\end{cases}$$

$$\int_{-a}^{b}|x|\,g(x)\,dx = \int_{-a}^{0}(-x)g(x) + \int_0^b x\,g(x)$$

$\boxed{L=2}$

$$a_n = \frac{1}{L}\int_{-L}^{L}\underbrace{f(x)}_{\text{even}}\underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}}dx$$

$$= 2\cdot\frac{1}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 f(x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 1\cdot\cos\left(\frac{n\pi x}{2}\right)dx$$

$u = x$, $du = dx$, $dv = \cos\left(\frac{n\pi x}{2}\right)$, $v = \sin\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}$

$$= \left.\frac{2}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)\right|_0^1 - \frac{2}{n\pi}\int_0^1\sin\left(\frac{n\pi x}{2}\right)dx + \left.\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right|_1^2$$

$$= \left.\frac{2}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)\right|_0^1 - \frac{2}{n\pi}\left[\left.-\cos\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}\right]\right|_0^1 + \left.\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right|_1^2$$

$$= \left.\frac{2}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)\right|_0^1 + \left.\left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi x}{2}\right)\right|_0^1 + \left.\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right|_1^2$$

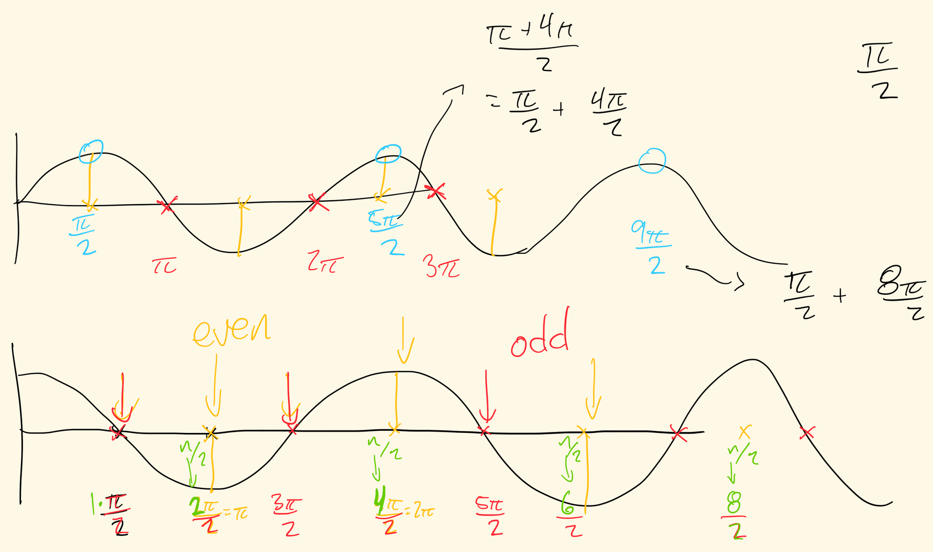
$$= \cancel{\frac{2}{n\pi}\cdot 1\cdot\sin\left(\frac{n\pi}{2}\right)} - 0 + \left(\frac{2}{n\pi}\right)^2\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] + \frac{2}{n\pi}\left[\sin(n\pi) - \cancel{\sin\left(\frac{n\pi}{2}\right)}\right]$$

$$= \left(\frac{2}{n\pi}\right)^2\cdot\begin{cases} -1 & n \text{ odd} \\ (-1)^{n/2} - 1 & n \text{ even}\end{cases}$$

$$a_n = \begin{cases} -\left(\frac{2}{n\pi}\right)^2, & n \text{ odd} \\ \left(\frac{2}{n\pi}\right)^2\left[(-1)^{n/2} - 1\right], & n \text{ even}\end{cases}$$

$\frac{n\pi}{2}$ for $n = 1, 2, 3, \dots$

$\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$



$\frac{\pi}{2}$

$$2K\pi + \frac{\pi}{2} = \frac{4K\pi + \pi}{2} = \frac{\pi(4K+1)}{2}$$

$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n \text{ odd} \\ (-1)^{n/2} & n \text{ even}\end{cases}$$

check: $n = 2$, $(-1)^{2/2} = (-1)^1 = (-1)$

$n = 4$, $(-1)^{4/2} = (-1)^2 = +1$

$n = 6$, $(-1)^{6/2} = (-1)^3 = (-1)$

So,

$f(x)$ as defined above is:

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{2}\right),$$

where a_n is defined by

$$a_n = \begin{cases} -\left(\frac{2}{n\pi}\right)^2, & n \text{ odd} \\ \left(\frac{2}{n\pi}\right)^2\left[(-1)^{n/2} - 1\right], & n \text{ even}\end{cases}$$

$n = 2K - 1$ (odd) $\qquad n = 2K$ (even)

$$\frac{3}{4} + \sum_{K=1}^{\infty} -\left(\frac{2}{(2K-1)\pi}\right)^2\cos\left(\frac{(2K-1)\pi x}{2}\right)$$

$$+ \sum_{K=1}^{\infty}\left(\frac{1}{K\pi}\right)^2\underbrace{\left[(-1)^K - 1\right]}\cos(K\pi x)$$

$$= \begin{cases} 0 & K \text{ is even} \\ -2 & K \text{ is odd}\end{cases}$$

set $K = 2j - 1$