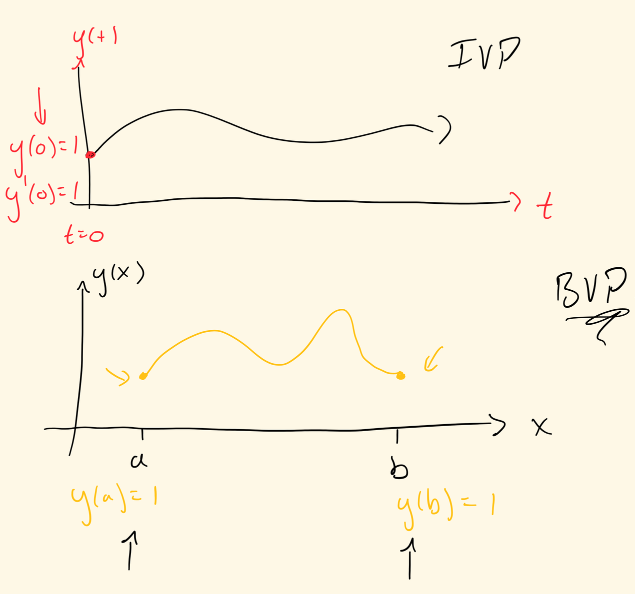
EL24 Lab 10 live

Thursday, April 1, 2021 4:00 PM

- Eigenvalue problems
- Fourier series.



Ex. $\begin{cases} y'' + \lambda y = 0 , & x \in (0,\pi) \\ y(0) = y'(\pi) = 0 \end{cases}$

3 Cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$

$\boxed{\lambda = 0}$: $y''(x) = 0 \to y'(x) = a \to y(x) = ax + b$

Apply bndy cond's: $y(0) = a\cdot 0 + b = \boxed{b = 0}$

$y'(x) = a$

$\hookrightarrow y'(\pi) = \boxed{a = 0}$

$\Rightarrow y(x) = 0$ (Trivial Sol$^{\underline{n}}$!)

$\lambda = 0$ is not an eigenvalue.

$\boxed{\lambda > 0}$: Set $\lambda = \mu^2 > 0$. Then,

$$y'' + \lambda y = y'' + \mu^2 y = 0$$

Char eq$^{\underline{n}}$: $r^2 + \mu^2 = 0$

$\to r^2 = -\mu^2$

$\hookrightarrow \boxed{r = \pm i\mu}$

$a y'' + b y' + c y = 0$

Char eq$^{\underline{n}}$: $a r^2 + \cancel{br}^{\,b=0} + c = 0$

$a = 1$, $c \to \mu^2$

$r = \pm i\mu$

$$y(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)$$

Apply bndy conds: $y(0) = C_1 \cos(0) + C_2 \cancel{\sin(0)}^{\,0}$

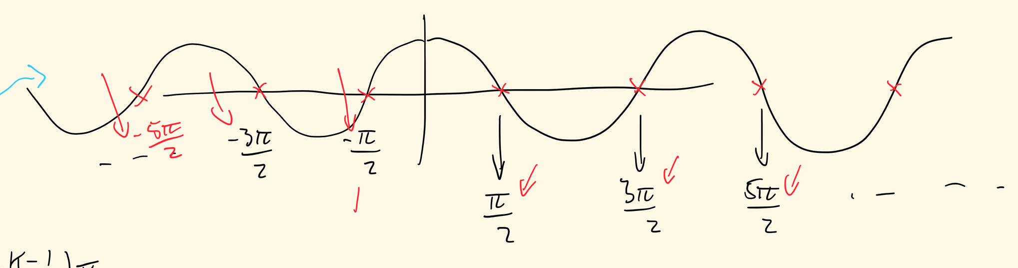
$= \boxed{C_1 = 0}$

$\hookrightarrow \boxed{y(x) = C_2 \sin(\mu x).}$

$y'(x) = C_2 \mu \cos(\mu x)$

$\hookrightarrow \boxed{y'(\pi) = C_2\, \mu\, \underbrace{\cos(\mu\pi)}_{=0}} = 0$

$\cancel{C_2 \neq 0}$ assume $C_2 \neq 0$, $\mu \neq 0$



$\cos(q) = 0$ whenever

$q = (2K-1)\cdot\frac{\pi}{2}$, $K \in \mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$

odd multiples of $\frac{\pi}{2}$

So, if $\cos(q) = 0$ when $q = \left(\frac{2K-1}{2}\right)\pi$

then, if $q = \mu\pi$, then

$\mu \cancel{\pi} = \left(\frac{2K-1}{2}\right)\cancel{\pi}$

$$\boxed{\mu_K = \left(\frac{2K-1}{2}\right), \quad K \in \mathbb{Z}}$$

For each eigenvalue, there is a distinct eigenfunction.

In this case,

$y_K(x) = \sin(\mu_K x)$

$\boxed{y_K(x) = \sin\left(\left(\frac{2K-1}{2}\right)x\right)}$ $\rightsquigarrow$ eigen functions

eigenvalues: $\boxed{\lambda_K = \mu_K^2 = \left(\frac{2K-1}{2}\right)^2}$ $\rightsquigarrow$ eigen values.

$\lambda < 0$: Set $\lambda = -\mu^2 < 0$. So,

$y'' + (-\mu^2) y = 0 \Rightarrow 1\cdot r^2 - \mu^2 = 0$

$\hookrightarrow r = \pm\mu.$

So,

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

Apply bndy cond.: $y(0) = c_1 e^0 + c_2 e^0 = \boxed{c_1 + c_2 = 0} \rightsquigarrow c_1 = -c_2$

$y'(x) = c_1 \mu e^{\mu x} - c_2 \mu e^{-\mu x}$

$y'(\pi) = c_1 \mu e^{\mu\pi} - c_2 \mu e^{-\mu\pi}$

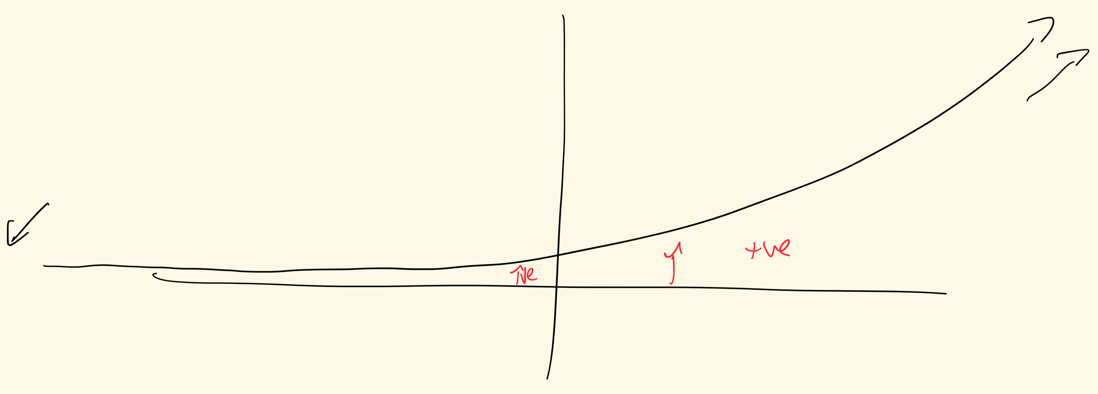
$= c_1 \mu e^{\mu\pi} + c_1 \mu e^{-\mu\pi}$

$= \underbrace{c_1}_{=0}\, \underbrace{\mu}_{\neq 0} \cdot \underbrace{\left(e^{\mu\pi} + e^{-\mu\pi}\right)}_{\cancel{=0}\ \neq 0} = 0$

only option is $c_1 = 0$.

$\Rightarrow c_2 = 0$

there are no negative eigenvalues.



$e^{\mu\pi} > 0$

$e^{-\mu\pi} > 0$

$\to e^{\mu\pi} + e^{-\mu\pi} > 0$

So, $y(x) = 0$, the trivial sol$^{\underline{n}}$.

Conclusion: Our problem has only Positive eigenvalues,

$\lambda_K = \left(\frac{2K-1}{2}\right)^2$,

we also have the corresponding eigenfunctions

$$y_K(x) = C_K \sin\left(\left(\frac{2K-1}{2}\right)x\right)$$

Fourier: f is periodic on $[-L, L]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

where:

$$\begin{cases} a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{cases}$$

Even: $f(x) = f(-x)$ ODD: $f(x) = -f(-x)$

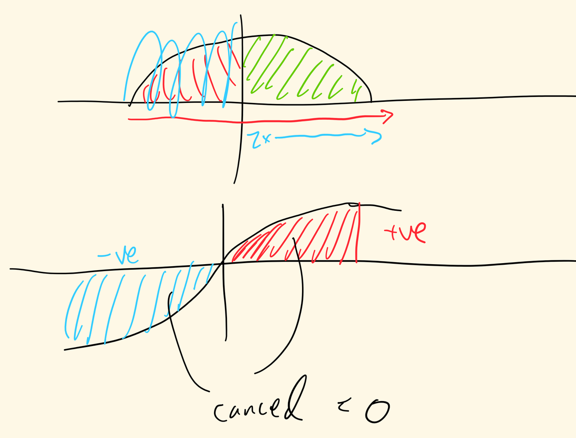
even × even = even, even × odd = odd,

odd × odd = even

2 Rules: On a symmetric interval:

$$\frac{1}{L}\int_{-L}^{L}(\text{even})\,dx = \frac{2}{L}\int_{0}^{L}(\text{even})\,dx$$

$$\frac{1}{L}\int_{-L}^{L}(\text{odd})\,dx = 0$$

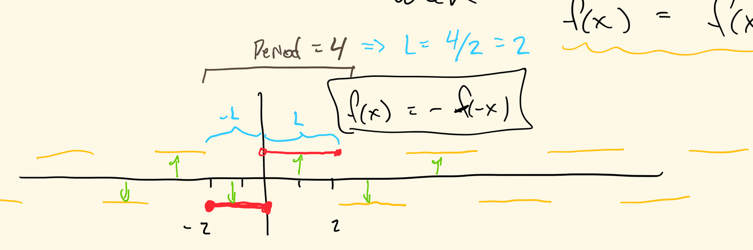


Ex. Find the F.S. of

$$f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ 1, & 0 < x \le 2 \end{cases}$$

where $f(x) = f(x+4)$

Picture:



To solve, note that: $f(x)$ is odd!

$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = 0$ (f is odd)

$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}}\cdot\underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}}}_{\text{odd}}dx = 0$ ($f\cdot\cos$ is odd)

$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}}\ \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}}}_{\text{even}}dx$

$= \frac{2}{L}\int_{0}^{L} f(x)\cdot\sin\left(\frac{n\pi x}{L}\right)dx$

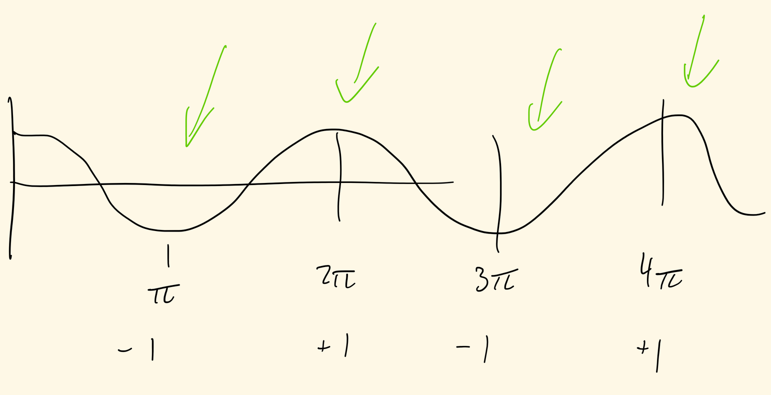
$= \frac{2}{2}\cdot\int_{0}^{2} 1\cdot\sin\left(\frac{n\pi x}{2}\right)dx$

$= -\cos\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}\Big|_{0}^{2}$

$= \frac{-2}{n\pi}\left[\cos(\pi n) - 1\right]$

$= \frac{-2}{n\pi}\left[(-1)^n - 1\right]$

$= \begin{cases} 0 & \text{for } n \text{ even} \\ \boxed{\frac{4}{n\pi} \quad \text{for } n \text{ odd.}} \end{cases}$



So, our sol$^{\underline{n}}$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$= \sum_{n=1}^{\infty}\left(b_n \sin\left(\frac{n\pi x}{2}\right)\right), \quad \text{write } n = 2K-1$$

$$\boxed{f(x) = \frac{4}{\pi}\sum_{K=1}^{\infty}\frac{1}{(2K-1)}\cdot\sin\left(\frac{(2K-1)\pi x}{2}\right)}$$

$$\cancel{f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{2}\right)}$$