EL21 lab 12 live

Thursday, April 15, 2021 9:59 AM

Ex. Solve

$$\begin{cases} \boxed{u_t = 3u_{xx}}, & 0<x<\pi,\ t>0 \\ u_x(0,t) = u_x(\pi,t) = 0, & t>0 \\ u(x,0) = f(x), & 0\le x\le\pi \end{cases} \longrightarrow \begin{aligned} u(x,t) &= X(x)T(t) \\ u_x(x,t) &= X'(x)T(t) \\ u_x(0,t) &= \underset{=0}{X'(0)}\cancel{T(t)} = 0, \quad t>0 \\ u_x(\pi,t) &= \underset{=0}{X'(\pi)}T(t) = 0, \quad t>0 \end{aligned}$$

First, assume $u(x,t) = X(x)\,T(t)$. Plug it in:

$$\frac{\partial u}{\partial t} = \frac{T'(t)X(x)}{T(t)X(x)} = \frac{3T(t)X''(x)}{T(t)X(x)} = \frac{\partial^2 u}{\partial x^2}$$

or:

$$\frac{T'(t)}{3T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

So:

$$\begin{cases} \text{I} & T'(t) + \lambda 3T(t) = 0 \quad \to \text{first order linear} \\ \text{II} & X''(x) + \lambda X(x) = 0 \quad \to \text{second order e.v. problem.} \end{cases}$$

I $\quad \frac{T'}{T} = -3\lambda \to (\ln(T))' = -3\lambda$

$\to \ln(T) = -3\lambda t + C$

$\to \boxed{T(t) = \tilde{C}e^{-3\lambda t}}$

II $\quad X'' + \lambda X = 0$ with $X'(0) = X'(\pi) = 0$

1. $\lambda<0 \Rightarrow$ exponential sol^{n}'s and find that the constants must be zero due to boundary conditions.

No e.v.'s <0 !

2. $\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = Ax + B.$

$X'(x) = A,\ X'(0) = X'(\pi) = \boxed{0 = A}$

B can be anything! Take $\boxed{B=1}$

$\lambda = 0$ is an e.v. with e.f. $X_0(x) = 1.$

3. $\lambda>0 \Rightarrow$ complex conjugates and

$X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

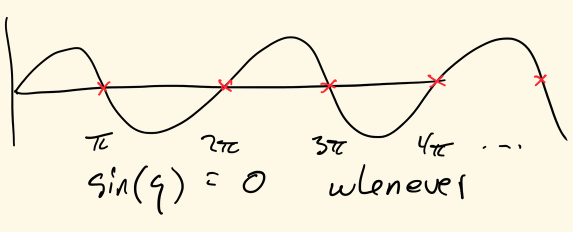
$X'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

$X'(0) = 0 + \underset{=0}{c_2}\underset{\neq 0}{\sqrt{\lambda}} = 0$

$\hookrightarrow X(x) = c_1\cos(\sqrt{\lambda}x)$

$X'(\pi) = -\underset{\neq 0}{c_1}\underset{\neq 0}{\sqrt{\lambda}}\,\underbrace{\sin(\sqrt{\lambda}\pi)}_{=0} = 0$

$\hookrightarrow \sin(\sqrt{\lambda}\pi) = 0$ whenever



$\sin(\xi) = 0$ whenever $\xi = n\cdot\pi$

$\sqrt{\lambda}\cdot\cancel{\pi} = n\cdot\cancel{\pi}$

$\hookrightarrow \lambda_n = n^2 \leadsto$

So, our eigenvalues are $\lambda_n = n^2$ with eigenfunctions $X_n(x) = C_n\cos(nx)$

Recall:

$T(t) = \tilde{C}e^{-3\lambda t}$

$\hookrightarrow T_n(t) = \tilde{C}e^{-3\lambda_n t} \to \tilde{C}=1$

$T_n(t) = \tilde{C}e^{-3n^2 t} \to \tilde{C}=1$

$X_n(x) = C_n\cos(nx)$

Then, $u(x,t) = X(x)T(t)$, but each $\underbrace{X_n(t)\,T_n(t) = C_n e^{-3n^2 t}\cos(nx)}$ is a solution!

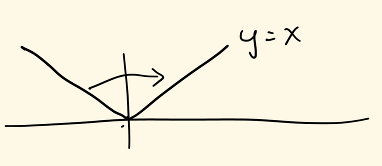
So,

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} C_n e^{-3n^2 t}\cos(nx)}$$

and $u(x,0) = f(x).$

$$u(x,0) = \sum_{n=0}^{\infty} C_n \overset{=1}{e^{0}}\cos(nx) = \sum_{n=0}^{\infty} C_n\cos(nx) = f(x)$$

on the interval $(0,\pi)$



So, suppose $\boxed{f(x) = x.}$

$$C_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$$

$$C_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx \qquad \begin{aligned} u &= x & dv &= \cos(nx) \\ du &= dx & v &= \tfrac{\sin(nx)}{n} \end{aligned}$$

$$= \frac{2}{\pi}\left[\frac{x\sin(nx)}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\right]$$

$$= \frac{2}{\pi}\left[\frac{x\sin(nx)}{n}\Big|_0^{\pi} - \frac{1}{n}\left(-\frac{\cos(nx)}{n}\right)\Big|_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\cancel{\frac{\pi\sin(n\pi)}{n}}^{\,0} - 0 + \frac{1}{n^2}\big(\underbrace{\cos(n\pi)}_{=(-1)^n} - 1\big)\right]$$

$$= \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

$$= \begin{cases} 0 & \text{when } n \text{ is even} \\ \frac{-4}{\pi n^2} & \text{when } n \text{ is odd} \end{cases}$$

Then,

$$x \sim \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n\cos(nx), \quad \text{write } n = 2k-1$$

$$= \frac{\pi}{2} + \sum_{k=1}^{\infty}\left(\frac{-4}{\pi(2k-1)^2}\cos((2k-1)x)\right)$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos((2k-1)x)$$

Finally, our "full" sol^{n} is:

$$\boxed{u(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{e^{-3(2k-1)^2 t}}{(2k-1)^2}\cos((2k-1)x)}$$

Ex. Solve:

$$\begin{cases} u_{tt} = 16u_{xx}, & x\in(0,\pi),\ t>0 \\ u(0,t) = u(\pi,t) = 0, & t>0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Suppose $u(x,t) = X(x)T(t)$, plug in:

$\hookrightarrow T''(t)X(x) = 16\,T(x)X''(x)$

$\hookrightarrow \frac{T''}{16T} = \frac{X''}{X} = -\lambda$

$$\begin{cases} T'' + 16\lambda T = 0 \\ X'' + \lambda X = 0 \end{cases}$$

$\lambda<0 \Rightarrow$ no sol^{n}'s

$\lambda=0 \Rightarrow$ no sol^{n}'s

$\lambda>0$: $\quad X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

Boundary cond: $X(0) = \boxed{c_1 + 0 = 0}$

$X(\pi) = c_2\sin(\sqrt{\lambda}\pi) = 0$

$\Rightarrow \sqrt{\lambda}\pi = n\cdot\pi$

$\Rightarrow \boxed{\lambda_n = n^2}$

e.f.'s $\quad X_n(x) = C_n\sin(nx)$

Back to $T(t)$: $\quad T_n'' + 16\cdot n^2 T_n = 0 \to r = \pm i\sqrt{16n^2} = \pm 4ni$

$\hookrightarrow T_n(t) = a_n\cos(4nt) + b_n\sin(4nt)$

$$u(x,t) = \sum_{n=1}^{\infty} C_n\sin(nx)\left[a_n\cos(4nt) + b_n\sin(4nt)\right]$$

assume $C_n = 1\ \forall n\geq 1.$

$$u(x,t) = \sum_{n=1}^{\infty}\sin(nx)\left[a_n\cos(4nt) + b_n\sin(4nt)\right]$$

$$u(x,0) = \sum_{n=1}^{\infty}\sin(nx)\left[a_n + 0\right]$$

$$= \boxed{\sum_{n=1}^{\infty} a_n\sin(nx) = f(x)} \leadsto a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty}\sin(nx)\left[a_n(-4n\sin(4nt)) + b_n(4n\cos(4nt))\right]$$

$$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty}\sin(nx)\left[0 + 4n\cdot b_n\right]$$

$$= \boxed{\sum_{n=1}^{\infty} 4nb_n\sin(nx) = g(x)}$$

set $d_n = b_n\cdot 4n$

$$= \sum_{n=1}^{\infty} d_n\sin(nx) = g(x) \leadsto d_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)\,dx$$