EL21 Lab 11 live

Thursday, April 8, 2021 10:02 AM

- Fourier Series: Given $f(x)$ defined on $(-L, L)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

where:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}}dx$$

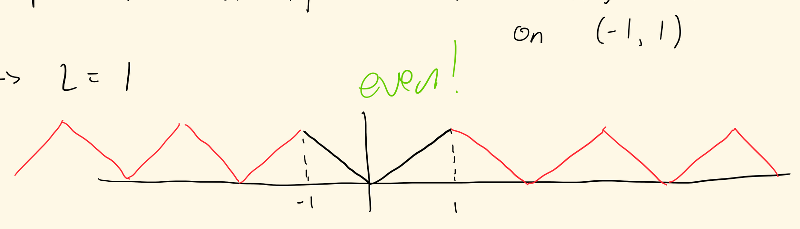
$$\int_{-L}^{L}(\text{even})\,dx = 2\int_{0}^{L}(\text{even})\,dx$$

$$\int_{-L}^{L}(\text{odd})\,dx = 0$$

Ex. Compute the F.S. expansion of $f(x) = |x|$ on $(-1, 1)$ $\sim$ $\boxed{f(x) = f(-x)}$ $|x| = |-x|$ $\Rightarrow$ even

$\rightarrow L = 1$

even!



$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{even}}\underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}}}_{\text{odd!}} dx = 0$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{1}\cdot 2\int_0^1 |x|\,dx = 2\int_0^1 x\,dx$$

$$= 2\cdot\frac{x^2}{2}\Big|_0^1 = \boxed{1}$$

$$a_n = \frac{1}{L}\int_{-L}^{L}\underbrace{\underbrace{f(x)}_{\text{even}}\cdot\underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}}}_{\text{even!}}dx = 2\cdot\frac{1}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{1}\int_0^1 x\cos\left(\frac{n\pi x}{1}\right)dx$$

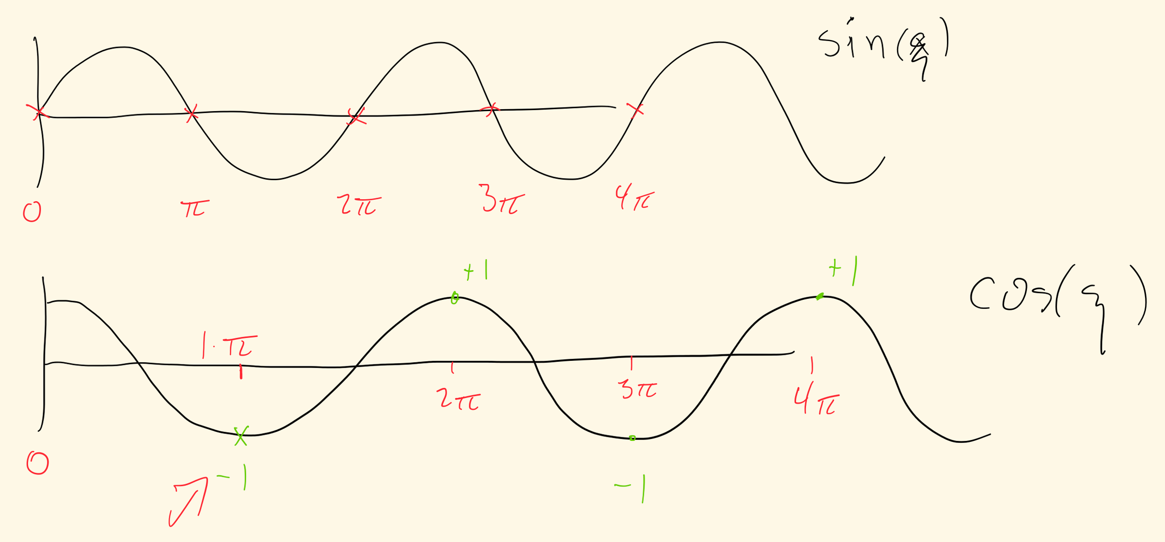
set $u = x$, $du = dx$, $dv = \cos(n\pi x)$, $v = \sin(n\pi x)\cdot\frac{1}{n\pi}$

$$= 2\left[\frac{x\sin(n\pi x)}{n\pi}\Big|_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi}\right]$$

$$= 2\left[\frac{x\sin(n\pi x)}{n\pi}\Big|_0^1\right] - \frac{2}{n\pi}\left[-\frac{\cos(n\pi x)}{n\pi}\Big|_0^1\right]$$

$$= 2\left[\cancel{\frac{1\cdot\sin(n\pi)}{n\pi}}\right] + \frac{2}{(n\pi)^2}\left[\cos(n\pi) - 1\right]$$

0



$$= 0 + \frac{2}{(n\pi)^2}\left[(-1)^n - 1\right]$$

$$\Rightarrow \quad a_n = \frac{2}{(n\pi)^2}\left[(-1)^n - 1\right]$$

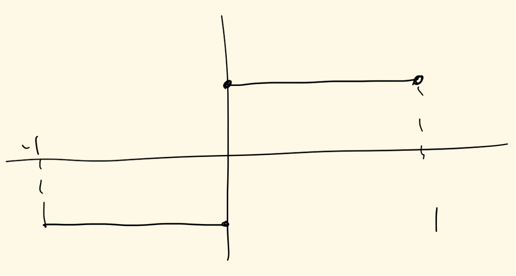
So,

$$f(x) = |x| = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\left[(-1)^n - 1\right]\cos(n\pi x)$$

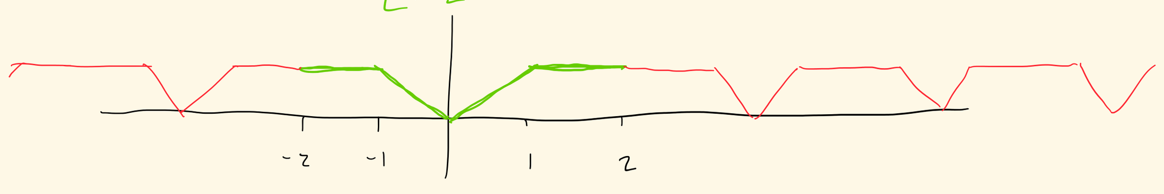
on $(-1, 1)$

Ex. Compute the F.S. expansion of:

$$f(x) = \begin{cases} 1 & -2 < x \le -1 \\ -x & -1 < x \le 0 \\ x & 0 < x \le 1 \\ 1 & 1 < x \le 2 \end{cases}$$



$L = 2$



even! $\Longrightarrow$ $\boxed{b_n = 0}$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = 2\cdot\frac{1}{2}\int_0^2 f(x)\,dx$$

$$= 1\cdot\int_0^1 x\,dx + 1\cdot\int_1^2 1\cdot dx$$

$$= \frac{x^2}{2}\Big|_0^1 + x\Big|_1^2 = \frac{1}{2} + 1$$

$$= \boxed{3/2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L}\underbrace{\underbrace{f(x)}_{\text{even}}\underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}}}_{\text{even}}dx$$

$$= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= 1\cdot\left[\int_0^1 f(x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= \int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 1\cdot\cos\left(\frac{n\pi x}{2}\right)dx$$

$u = x$, $du = dx$, $dv = \cos\left(\frac{n\pi x}{2}\right)$, $v = \sin\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}$

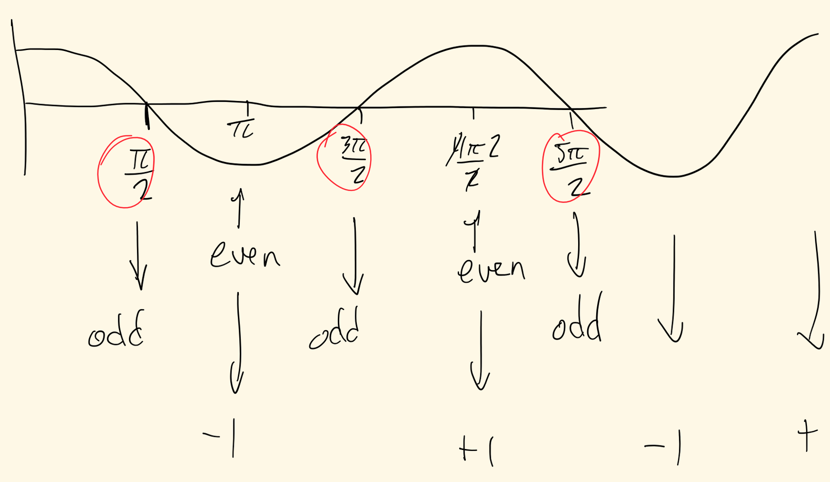
$$= x\cdot\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 - \int_0^1\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)dx + \int_1^2\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 - \frac{2}{n\pi}\left(-\cos\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}\Big|_0^1\right) + \sin\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}\Big|_1^2$$

$$= \frac{2}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 + \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1 + \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2$$

$$= \cancel{\frac{2}{n\pi}1\cdot\sin\left(\frac{n\pi}{2}\right)} + \left(\frac{2}{n\pi}\right)^2\left[\underbrace{\cos\left(\frac{n\pi}{2}\right)}_{\text{odd}\ =0} - 1\right] + \frac{2}{n\pi}\left[\underbrace{\sin(n\pi)}_{=0} - \cancel{\sin\left(\frac{n\pi}{2}\right)}\right]$$

$$\Rightarrow \quad a_n = \begin{cases} -\left(\frac{2}{n\pi}\right)^2, & n \text{ odd} \\ \left(\frac{2}{n\pi}\right)^2\left[(-1)^{n/2} - 1\right], & n \text{ even}\end{cases}$$



$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n \text{ odd} \\ (-1)^{n/2}, & n \text{ even}\end{cases}$$

and so

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{2}\right)$$

$$= \frac{3}{4} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{2}\right)$$

where

$$\boxed{a_n = \begin{cases} -\left(\frac{2}{n\pi}\right)^2, & n \text{ odd} \\ \left(\frac{2}{n\pi}\right)^2\left[(-1)^{n/2} - 1\right], & n \text{ even}\end{cases}}$$

Partial Differential Eq='s

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = f(x) \rightarrow x \text{ is only variable}$$

$$\dot{y} + y = 7, \quad \dot{y} = \frac{dy}{dt}$$

Has derivatives & multiple variables!

Classic Example:

$$\begin{cases} \boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}}, & x \in (-L, L),\ t > 0 \\ u(-L,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

IBVP $\rightarrow$ initial boundary value problem