Lab 9 Prep

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Laplace Review for midterm:

- delta & step functions
- solve a problem with them
- convolution theorem + solve integro-diff^n eq$^=$.

<u>Delta Function</u>: Has a "sifting" property + unit integral:

(i.) $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$, and

(ii.) $\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = f(t_0)$

For Laplace: $\mathcal{L}\{\delta(t-c)\} = e^{-cs}$.
To invert $e^{-cs}F(s)$, use that: $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)\cdot f(t-c)$,
where $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

<u>How to use this</u>: Solve: $\begin{cases} 2y'' + y' + 2y = \delta(t-5) \\ y(0) = 0,\ y'(0) = 0 \end{cases}$

1. Take Laplace of both sides: (solve for $Y(s)$)

$$2s^2Y(s) + sY(s) + 2Y(s) = e^{-5s}$$
$$\hookrightarrow Y(s) = \frac{e^{-5s}}{2s^2+s+2} = \frac{e^{-5s}}{2}\cdot\frac{1}{(s^2+\frac{s}{2}+1)}$$

2. Massage into something familiar: complete the square (no partial fractions)

$$s^2 + s/2 + 1 = (s+1/4)^2 - 1/16 + 1$$
$$= (s+1/4)^2 + 15/16$$

So, $Y(s) = \frac{e^{-5s}}{2}\cdot\frac{1}{(s+1/4)^2 + (\sqrt{\frac{15}{16}})^2}$

3. Invert $Y(s)$. Notice that $Y(s)$ looks like $\frac{1}{2}e^{-cs}F(s)$,
where $c = 5$ and $F(s) = \frac{1}{(s+1/4)^2 + (\sqrt{\frac{15}{16}})^2}$

So, use item 13: $\mathcal{L}^{-1}(e^{-cs}F(s)) = u_c(t)\,f(t-c)$.

So, what is $\mathcal{L}^{-1}\{F(s)\}$? Looks like shifted sin,
where shift $= -1/4$. So, write $F(s)$ as:

$$F(s) = \frac{1}{(s+1/4)^2 + (\sqrt{\frac{15}{16}})^2} = \left(\sqrt{\tfrac{15}{16}}\right)^{-1}\cdot\frac{\sqrt{\frac{15}{16}}}{(s-(-\frac{1}{4}))^2 + (\sqrt{\frac{15}{16}})^2}$$

Hence, $\mathcal{L}^{-1}\{F(s)\} = e^{-\frac{1}{4}t}\cdot\sin(\frac{\sqrt{15}}{4}t)$

Thus,

$$\boxed{\mathcal{L}^{-1}\{\tfrac{1}{2}e^{-5s}F(s)\} = \frac{1}{2}u_5(t)\cdot e^{-\frac{1}{4}t}\sin(\tfrac{\sqrt{15}}{4}t)\cdot\frac{4}{\sqrt{15}} = \frac{2}{\sqrt{15}}u_5(t)\,e^{-\frac{1}{4}(t-5)}\sin(\tfrac{\sqrt{15}}{4}(t-5))}$$

<u>Ex.</u> Solve:

$$\begin{cases} y'' + 3y' + 2y = \delta(t-5) + u_{10}(t) \\ y(0) = 0 \\ y'(0) = 1/2 \end{cases}$$

1. Laplace of both sides: $\mathcal{L}\{y'' + 3y' + 2y\} = (s^2Y(s) - s\cancel{y(0)}^{0} - y'(0))$
$+ 3(sY(s) - \cancel{y(0)}^{0})$
$+ 2Y(s)$
$= (s^2 + 3s + 2)Y(s) - 1/2$

$$\mathcal{L}\{\delta(t-5) + u_{10}(t)\} = e^{-5s} + \frac{e^{-10s}}{s}$$

2. Solve for $Y(s)$: $Y(s) = \frac{1}{(s^2+3s+2)}\left[\frac{1}{2} + e^{-5s} + \frac{e^{-10s}}{s}\right]$

3. Invert: Partial fractions on $\frac{1}{s(s^2+3s+2)}$: $\frac{1}{s}\cdot\frac{1}{s^2+3s+2} = \frac{As+B}{s^2+3s+2} + \frac{C}{s} = \frac{(As+B)s + C(s^2+3s+2)}{(\quad)} = \underbrace{(A+C)}_{=0}s^2 + \underbrace{(B+3C)}_{=0}s + \underbrace{2C}_{=1}$

$\Rightarrow C = 1/2$,
$\Rightarrow A = -1/2$,
$\Rightarrow B = -3/2$

Complete square on $s^2 + 3s + 2 = (s+3/2)^2 - \frac{9}{4} + 2$
$= (s+3/2)^2 - 1/4$

$\frac{6}{4} - \frac{3}{4} = \frac{3}{4}$

So,
$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\cdot\frac{1}{s} - \left(\frac{1}{2}\frac{s}{s^2+3s+2} + \frac{3}{2}\frac{1}{s^2+3s+2}\right)$$
$$= \frac{1}{2}\cdot\frac{1}{s} - \left(\frac{1}{2}\cdot\frac{(s+3/2)}{(s+3/2)^2 - 1/4} + \left(\frac{3}{2} - \frac{3}{4}\right)\cdot\frac{1}{(s+3/2)^2 - 1/4}\right)$$
$$= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\frac{(s-(-3/2))}{(s-(-3/2))^2 - 1/4} - \frac{3}{4}\cdot\frac{1}{(s-(-3/2))^2 - (1/2)^2}$$
$$= \underbrace{\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\frac{(s-(-3/2))}{(s-(-3/2))^2 - 1/4} - \frac{3}{4}\cdot 2\cdot\frac{1/2}{(s-(-3/2))^2 - (1/2)^2}}_{F(s)}$$

So, $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2} - \frac{1}{2}\cdot e^{-3/2t}\cos(\frac{1}{2}t) - \frac{3}{2}e^{-3/2t}\cdot\sin(\frac{1}{2}t)$
$= \frac{1}{2}\left(1 - e^{-\frac{3}{2}t}[\cos(\frac{1}{2}t) + 3\sin(\frac{1}{2}t)]\right) = f(t)$.

So, $\mathcal{L}^{-1}\{e^{-10s}F(s)\} = u_{10}\cdot f(t-10)$
$= u_{10}(t)\cdot\frac{1}{2}\cdot\left(1 - e^{-\frac{3}{2}(t-10)}[\cos(\frac{1}{2}(t-10)) + 3\sin(\frac{1}{2}(t-10))]\right)$

Finally,
$$\frac{1}{s^2+3s+2} = \frac{1}{(s-(-3/2))^2 - (1/2)^2} = 2\cdot\frac{1/2}{(s-(-3/2))^2 - (\frac{1}{2})^2}$$
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = e^{-3/2t}\cdot\sin(\tfrac{1}{2}t).$$

So, $\mathcal{L}^{-1}\{\frac{1}{2}\cdot\frac{1}{s^2+3s+2}\} = \frac{1}{2}e^{-3/2t}\sin(\frac{1}{2}t)$, and
$\mathcal{L}^{-1}\{e^{-5s}\cdot\frac{1}{s^2+3s+2}\} = u_5(t)\cdot f(t-5)$
$= u_5(t)\,e^{-\frac{3}{2}(t-5)}\sin(\frac{1}{2}(t-5))$.

So, $\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}e^{-\frac{3}{2}t}\sin(\frac{1}{2}t) + u_5(t)\,e^{-\frac{3}{2}(t-5)}\sin(\frac{1}{2}(t-5))$
$+ u_{10}(t)\cdot\frac{1}{2}\cdot\left[1 - e^{-\frac{3}{2}(t-10)}(\cos(\frac{1}{2}t) + 3\sin(\frac{1}{2}t))\right]$

Convolution Theorem: $\mathcal{L}\{f*g(t)\}(s) = F(s)\cdot G(s)$.

Ex. Invert $H(s) = \frac{a}{s^2(s^2+a^2)}$.

Write as: $H(s) = \frac{1}{s^2}\cdot\frac{a}{(s^2+a^2)} =: F(s)\cdot G(s)$.

Then, $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{1!}{s^{1+1}}\} = t$

$\mathcal{L}^{-1}\{G(s)\} = \sin(at)$ (Rule 5)

So, $\mathcal{L}^{-1}\{H(s)\} = f*g = \int_0^t f(t-\tau)g(\tau)\,d\tau$
$= \int_0^t (t-\tau)\sin(a\tau)\,d\tau$
$= t\int_0^t \sin(a\tau) - \int_0^t \tau\sin(a\tau)$ $u = \tau$, $du = 1d\tau$ $dv = \sin(a\tau)$, $v = -\frac{1}{a}\cos(a\tau)$
$= t\cdot -\frac{1}{a}\cos(a\tau)\Big|_0^t - \left[-\frac{\tau}{a}\cos(a\tau)\Big|_0^t + \frac{1}{a}\int_0^t\cos(a\tau)\right]$
$= -\cancel{\frac{t\cos(at)}{a}} + \frac{t}{a} - \left[-\cancel{\frac{t\cos(at)}{a}} + 0 + \frac{1}{a^2}\sin(a\tau)\Big|_0^t\right]$
$= -\frac{\sin(at)}{a^2} + \frac{t}{a} = \frac{1}{a^2}(at - \sin(at))$

Integro diff$^=$ eq$^=$: solve $\varphi(t) + \int_0^t (t-\xi)\varphi(\xi)\,d\xi = \sin(2t)$

1. Laplace both sides: $\mathcal{L}\{\varphi\} = \Phi(s)$,
$\mathcal{L}\{f*g\} = F(s)\cdot G(s)$, where $f(t) = t \Rightarrow \mathcal{L}\{t\} = \frac{1}{s^2}$, $g(t) = \varphi(t)$
$\Rightarrow \mathcal{L}\{\int_0^t (t-\xi)\varphi(\xi)\,d\xi\}(s) = \frac{1}{s^2}\cdot\Phi(s)$
$\mathcal{L}\{\sin(2t)\}(s) = \frac{2}{s^2+2^2}$

Then,
$$\Phi(s) + \frac{1}{s^2}\Phi(s) = \frac{2}{s^2+2^2}$$
$$\hookrightarrow \Phi(s)\left[1 + \frac{1}{s^2}\right] = \frac{2}{s^2+2^2}$$
$$\hookrightarrow \Phi(s)\left[\frac{s^2+1}{s^2}\right] = \frac{2}{(s^2+2^2)}$$
$$\hookrightarrow \Phi(s) = \frac{2s^2}{(s^2+1)\cdot(s^2+2^2)} = \underbrace{\frac{2s}{s^2+1^2}}_{2\cos(t)}\cdot\underbrace{\frac{s}{(s^2+2^2)}}_{\cos(2t)}$$

So, use conv. thm. again:
$\mathcal{L}^{-1}\{\Phi(s)\} = \mathcal{L}^{-1}\{F(s)\cdot G(s)\} = f*g$, where $f(t) = 2\cos(t)$, $g(t) = \cos(2t)$

$2I = \int_0^t 2\cos(t-\tau)\cos(2\tau)\,d\tau$ $u = \cos(t-\tau)$, $du = +\sin(t-\tau)$ $dv = \cos(2\tau)$, $v = \frac{1}{2}\sin(2\tau)$
$= \frac{2}{2}\sin(2\tau)\cos(t-\tau)\Big|_0^t - \frac{2}{2}\int_0^t \sin(t-\tau)\sin(2\tau)\,d\tau$
$= \sin(2t) - \int_0^t \sin(t-\tau)\sin(2\tau)\,d\tau$ $u = \sin(t-\tau)$, $du = +\cos(t-\tau)$ $dv = \sin(2\tau)$, $V = -\frac{1}{2}\cos(2\tau)$
$= \sin(2t) - \left[-\frac{1}{2}\sin(t-\tau)\cos(2\tau)\Big|_0^t + \frac{1}{2}\int\cos(t-\tau)\cos(2\tau)\,d\tau\right]$
$= \sin(2t) - \left[\frac{1}{2}\sin(t) + \frac{1}{2}I\right]$

$\hookrightarrow \underbrace{\left[2 - \frac{1}{2}\right]}_{\frac{3}{2}} I = \sin(2t) - \frac{1}{2}\sin(t)$

$\Rightarrow I = \frac{2}{3}\sin(2t) - \frac{1}{3}\sin(t)$