EL24 Lab 9 live

Thursday, March 25, 2021 3:58 PM

Ex. $\begin{cases} 2y'' + y' + 2y = \delta(t-5) = \delta_5(t) \\ y(0) = y'(0) = 0 \end{cases}$

1. Take Laplace of both sides: $\mathcal{L}\{2y'' + y' + 2y\}$
$= 2s^2 Y(s) + sY(s) + 2Y(s)$
$= (2s^2 + s + 2) Y(s)$
$\mathcal{L}\{\delta(t-5)\} = e^{-5s}$ (#17 on table)

2. Solve for $Y(s)$: $Y(s) = \dfrac{e^{-5s}}{(2s^2+s+2)} = e^{-5s} \cdot \dfrac{1}{(2s^2+s+2)}$ $\equiv F(s)$

3. Invert $Y(s)$: Use #13 on the table: $e^{-cs}F(s)$ if we know $\mathcal{L}^{-1}\{F(s)\}$ on its own.

3a. Invert $F(s)$, then $\mathcal{L}^{-1}\{e^{-5s}F(s)\} = u_5(t) \cdot f(t-5)$

So, turn it into something familiar!

$\dfrac{b}{(s-a)^2 + b^2}$

$F(s) = \dfrac{1}{(2s^2+s+2)} = \dfrac{1}{2(s^2 + \frac{s}{2} + 1)}$, now complete square!

$s^2 + \frac{s}{2} + 1 = (s + \frac{1}{4})^2 - \frac{1}{16} + 1 = (s+\frac{1}{4})^2 + \frac{15}{16}$

$s^2 + \frac{s}{2} + 1$
$= (s+1/4)^2 - (\frac{1}{4})^2 + 1$
$= (s+1/4)^2 - \frac{1}{16} + \frac{16}{16}$
$= (s+1/4)^2 + \frac{15}{16}$

(item #9)

$= \frac{1}{2} \cdot \dfrac{1}{(s+1/4)^2 + (\sqrt{\frac{15}{16}})^2}$

$= \frac{1}{2} \left(\sqrt{\tfrac{15}{16}}\right)^{-1} \cdot \dfrac{\sqrt{\frac{15}{16}}}{(s+1/4)^2 + (\sqrt{\frac{15}{16}})^2}$

$= \frac{1}{2} \left(\sqrt{\tfrac{15}{16}}\right)^{-1} \cdot \dfrac{b}{(s-a)^2 + b^2}$, where: $a = -1/4$, $b = \sqrt{\frac{15}{16}}$

So, from item #9 in table:

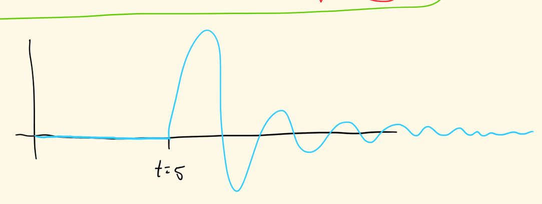
$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}\left(\sqrt{\tfrac{15}{16}}\right)^{-1} \cdot e^{-\frac{1}{4}t} \sin\left(\sqrt{\tfrac{15}{16}}\, t\right)$

$$f(t) = \frac{2}{\sqrt{15}} \cdot e^{-\frac{1}{4}t} \cdot \sin\left(\frac{\sqrt{15}}{4} t\right)$$

3b. So, from item #13 in the table:

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{e^{-5s}F(s)\}$
$= u_5(t) \cdot f(t-5)$

$$y(t) = u_5(t) \cdot \frac{2}{\sqrt{15}} \cdot e^{-\frac{1}{4}(t-5)} \cdot \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)$$

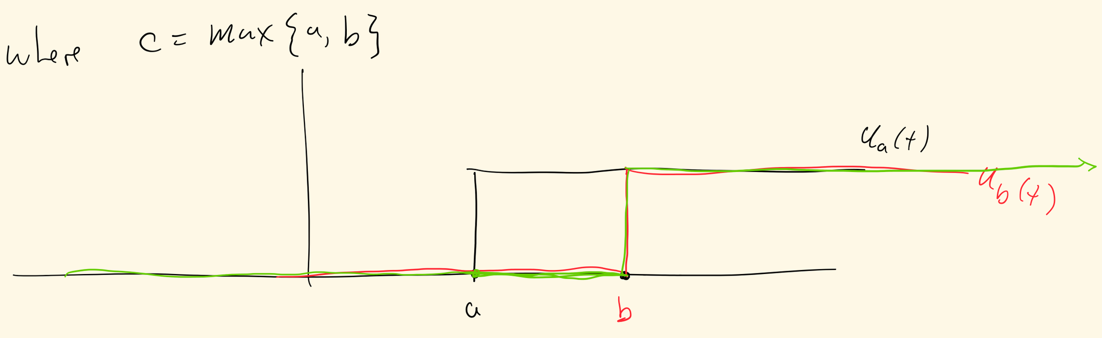


$\int_{-\infty}^{\infty} \delta(t-c) f(t)\,dt = f(c)$

$\delta(t-5) \cdot g(t)$

$\mathcal{L}\{\delta(t-5) \cdot g(t)\} = \int_0^\infty e^{-st} \delta(t-5) \cdot g(t)\,dt$
$= \int_0^\infty (e^{-st} \cdot g(t))\, \delta(t-5)\,dt$ — $f(t)$, 5
$= f(5) = e^{-5s} \cdot g(5)$

$u_a(t) \cdot u_b(t) = u_c(t)$,
where $c = \max\{a, b\}$



Ex. $\begin{cases} y'' + 3y' + 2y = \delta(t-5) + u_{10}(t) \\ y(0) = 0 \\ y'(0) = 1/2 \end{cases}$

1. Laplace of both sides: $\mathcal{L}\{y'' + 3y' + 2y\} = (s^2 Y(s) - sy(0) - y'(0)) + 3(sY(s) - y(0)) + 2Y(s)$ [$y(0) \to 0$, $y'(0) = 1/2$]
$= (s^2 + 3s + 2) Y(s) - 1/2$

$\mathcal{L}\{\delta(t-5) + u_{10}(t)\} = e^{-5s} + \dfrac{e^{-10s}}{s}$

2. Solve for $Y(s)$: $Y(s) = \left(\frac{1}{2} + e^{-5s} + \dfrac{e^{-10s}}{s}\right) \cdot \dfrac{1}{(s^2+3s+2)}$

3. Invert $Y(s)$:
$Y(s) = \left(\frac{1}{2} + e^{-5s}\right) \cdot \dfrac{1}{(s^2+3s+2)}$ ⇝ looks like $\left(\frac{1}{2} + e^{-5s}\right)F(s)$ "easy"
$+ e^{-10s} \cdot \dfrac{1}{s(s^2+3s+2)}$ ⇝ looks like $e^{-10s} \cdot G(s)$, where G needs partial fractions

Focus on: $F(s) = \dfrac{1}{s^2+3s+2}$ and
$G(s) = \dfrac{1}{s(s^2+3s+2)}$

Notice: $s^2 + 3s + 2 = (s+2)(s+1)$,

write: $\dfrac{1}{(s+2)(s+1)} = \dfrac{A}{s+2} + \dfrac{B}{s+1}$
$= \dfrac{A(s+1) + B(s+2)}{(\;)(\;)}$
$= \dfrac{(A+B)s + (A+2B)}{(\;)(\;)}$ ⟶ $A + B = 0$ ⇝ $A = -B$; $A + 2B = 1$ ↳ $-B + 2B = 1$ ⟹ $B = 1$ ⟹ $A = -1$

$$F(s) = \frac{-1}{s+2} + \frac{1}{s+1}$$

$G(s) = \dfrac{1}{s(s+1)(s+2)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+2}$
⟹ $A = B = 1/2$
$C = -1$

$$G(s) = \frac{1}{2} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{1}{(s+1)} - \frac{1}{s+2}$$

$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{-1}{s-(-2)} + \dfrac{1}{s-(-1)}\right\} = -e^{-2t} + e^{-t} = f(t)$

$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{2} \cdot \frac{1}{s} + \frac{1}{2} \cdot \dfrac{1}{(s-(-1))} - \dfrac{1}{s-(-2)}\right\} = \frac{1}{2} + \frac{1}{2}e^{-t} - e^{-2t} = g(t)$

$\mathcal{L}^{-1}\left\{\left(\frac{1}{2} + e^{-5s}\right)F(s)\right\} = \frac{1}{2}\mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{e^{-5s}F(s)\}$
$= \frac{1}{2} \cdot f(t) + u_5(t) \cdot f(t-5)$
$= \frac{1}{2} \cdot \left(e^{-t} - e^{-2t}\right) + u_5(t) \cdot \left(e^{-(t-5)} - e^{-2(t-5)}\right)$

$\mathcal{L}^{-1}\{e^{-10s}G(s)\} = u_{10}(t) \cdot g(t-10)$
$= u_{10}(t) \cdot \left[\frac{1}{2} + \frac{1}{2}e^{-(t-10)} - e^{-2(t-10)}\right]$

Put all together: $\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\left(\frac{1}{2} + e^{-5s}\right)F(s) + e^{-10s}G(s)\right\}$

$$y(t) = \frac{1}{2}\left(e^{-t} - e^{-2t}\right) + u_5(t)\left(e^{-(t-5)} - e^{-2(t-5)}\right) + u_{10}(t)\left[\frac{1}{2} + \frac{1}{2}e^{-(t-10)} - e^{-2(t-10)}\right]$$

Ex. $\phi(t) + \int_0^t (t-\tau)\,\phi(\tau)\,d\tau = \sin(2t)$
[$(t-\tau)$: $f(t-\tau)$ ↳ $f(t) = t$; $\phi(\tau)$: $g(\tau)$]

1. Laplace of both sides: $\mathcal{L}\{\phi(t)\} = \Phi(s)$
$\mathcal{L}\{\sin(2t)\} = \dfrac{2}{s^2+2^2}$
$\mathcal{L}\{f * g\} = F(s) \cdot G(s)$,
$= \dfrac{1}{s^2} \cdot \Phi(s)$

$\mathcal{L}\{f\} = \mathcal{L}\{t\} = \dfrac{1!}{s^{1+1}} = \dfrac{1}{s^2}$
$\mathcal{L}\{\phi(t)\} = \Phi(s)$

$h(t) = 2\cos(t)$
$j(t) = \cos(2t)$
$h * g = \int_0^t h(t-\tau) \cdot j(\tau)\,d\tau$
↳ $= \int_0^t 2\cos(t-\tau)\cos(2\tau)$
$= \int_0^t 2\cos(\tau)\cos(2(t-\tau))\,d\tau$

2. Solve for $\Phi(s)$: $\Phi(s)\left[1 + \dfrac{1}{s^2}\right] = \dfrac{2}{s^2+2^2}$
↳ $\Phi(s)\left[\dfrac{s^2+1}{s^2}\right] = \dfrac{2}{s^2+2^2}$
↳ $\Phi(s) = \dfrac{2s^2}{(s^2+1)(s^2+2^2)}$

3. Invert: $\Phi(s) = \dfrac{2s}{(s^2+1^2)} \cdot \dfrac{s}{(s^2+2^2)}$ ⟶ $\mathcal{L}^{-1}\{H(s)\} = 2\cos(t)$, $\mathcal{L}^{-1}\{J(s)\} = \cos(2t)$
(first factor ↓ $H(s)$, second factor ↓ $J(s)$)

⟹ $\mathcal{L}^{-1}\{\Phi(s)\} = \phi(t) = h * j$, where

$$\phi(t) = \int_0^t 2\cos(t-\tau) \cdot \cos(2\tau)\,d\tau$$

$\phi(0) = \int_0^0 2\cos(t-\tau)\cos(2\tau)\,d\tau = 0$

$\phi'(t) = \dfrac{d}{dt}\left(\int_0^t 2\cos(t-\tau)\cos(2\tau)\,dt\right)$
$= 2\cos(t-\tau)\cos(2\tau)\Big|_{\tau=t}$
$= 2\cos(0)\cos(2t)$
$= 2\cos(2t)$

$\phi'(0) = 2\cos(0) = 2$

$\dfrac{d}{dx}\int^x f(t)\,dt = f(x)$

The initial cond's must be: $\phi(0) = 0$, $\phi'(0) = 2$