EL24 Lab 8 live

Thursday, March 18, 2021 3:57 PM

- Solving IVP's using Laplace
- Convolutions.

General Strategy:

$$\begin{tikzcd}
\text{IVP} \arrow[r, "\text{solve.}"] \arrow[d, "1."'] & \text{Sol}^{n} \\
\text{L-T} \arrow[r, "\text{solve for } Y(s)", "2."'] & Y(s) \arrow[u, "3.\ \text{invert back to } y(t)"']
\end{tikzcd}$$

Ex. $\begin{cases} y'' - 2y' + 5y = -8e^{-t}, & t>0 \\ y(0) = 2 \\ y'(0) = 12 \end{cases}$

$\mathcal{L}\{y''\} = s^2 Y(s) - s\cdot y(0) - 1\cdot y'(0)$

$\mathcal{L}\{-2y'\} = -2(sY(s) - y(0))$

$\mathcal{L}\{5y\} = 5\cdot Y(s)$

$\mathcal{L}\{-8e^{-t}\} = -8\cdot \frac{1}{s+1}$

$\rightarrow \mathcal{L}\{eq^{=}\} = \boxed{\begin{array}{l} s^2Y(s) - s\cdot 2 - 12 \\ -2sY(s) + 2\cdot 2 \\ +5Y(s) \end{array} = \boxed{-\frac{8}{s+1}}}$ 1

Step 2: Solve for $Y(s)$

$$Y(s)[s^2 - 2s + 5] = -\frac{8}{s+1} + 2s + 8$$
$$= \frac{-\cancel{8} + (2s+\cancel{8})(s+\cancel{1})}{s+1}$$
$$= \frac{2s^2 + 10s}{s+1}$$

$$\boxed{Y(s) = \frac{2s^2+10s}{(s+1)(s^2-2s+5)}}$$

Step 3a. Change the form of $Y(s)$ into something familiar.

Partial Fractions:

$$Y(s) = \underbrace{\frac{3s+5}{s^2-2s+5}}_{\text{looks like sin/cosine.}} - \underbrace{\frac{1}{s+1}}_{\text{easy} \to \text{exp.}} =: F_1(s) + F_2(s).$$

$$\frac{2s^2 + 10s + 0}{(s+1)(s^2-2s+5)} = \frac{As+B}{s^2-2s+5} + \frac{C}{s+1}$$
$$= \frac{(As+B)(s+1) + C(s^2-2s+5)}{(s+1)(s^2-2s+5)}$$
$$= s^2(A+C) + s(A+B-2C) + (B+5C)$$

$A + C = 2 \rightsquigarrow C = 2-A$

$A + B - 2C = 10$

$B + 5C = 0 \quad B = -5C$
$= -5(2-A)$
$= 5A - 10$

$\hookrightarrow A + (5A-10) - 2(2-A) = 10$

$\hookrightarrow A(1+5+2) = 10 + 4 + 10$

$8A = 24$

$A = 3$

$\hookrightarrow C = 2 - A = 2 - 3 = -1$

$\hookrightarrow B = 5\cdot 3 - 10 = 5$

$\boxed{\begin{array}{l} A = 3 \\ B = 5 \\ C = -1 \end{array}}$

Step 3b. Complete the square: $s^2 - 2s + 5$
$= (s-1)^2 - 1 + 5$
$= (s-1)^2 + 2^2$

$$F_1(s) = \frac{3s+5}{(s-1)^2+2^2} = \frac{3(s-1) + 3 + 5}{(s-1)^2 + 2^2}$$
$$= \frac{3(s-1)}{(s-1)^2+2^2} + \frac{8}{(s-1)^2+2^2}$$
$$= 3\cdot\underbrace{\frac{s-1}{(s-1)^2+2^2}}_{\text{shifted cosine}} + 4\cdot\underbrace{\frac{2}{(s-1)^2+2^2}}_{\text{shifted sin}}$$

$\mathcal{L}^{-1}\left\{3\frac{s-1}{(s-1)^2+2^2}\right\} = 3e^{t}\cos(2t)$

$\mathcal{L}^{-1}\left\{4\frac{2}{(s-1)^2+2^2}\right\} = 4e^{t}\cdot\sin(2t)$

$\Bigg\}\ \mathcal{L}^{-1}\{F_1(s)\} = e^{t}(3\cos(2t) + 4\sin(2t))$

$\mathcal{L}^{-1}\left\{\frac{-1}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s-(-1)}\right\} = -e^{-t}.$

Thus: $\mathcal{L}^{-1}\{Y(s)\} = \boxed{\begin{array}{l} y(t) \\ = e^{t}(3\cos(2t) + 4\sin(2t)) - e^{-t} \end{array}}$

Ex. $\begin{cases} y'' + 2y' + 2y = u(t-2\pi) - u(t-4\pi) & (= u_{2\pi}(t) - u_{4\pi}(t)) \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$

Recall: $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

1. $\mathcal{L}\{y'' + 2y' + 2y\} = s^2Y(s) - sy(0) - y'(0)$
$2(sY(s) - y(0))$
$+2Y(s)$
$= Y(s)[s^2 + 2s + 2] - (s+3)$

$\mathcal{L}\{u(t-a)\} = \mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s}$

$\mathcal{L}\{u(t-2\pi)\} = \mathcal{L}\{u_{2\pi}(t)\} = \frac{e^{-2\pi s}}{s}$

$\mathcal{L}\{u_{4\pi}(t)\} = \frac{e^{-4\pi s}}{s}$

2. Solve for $Y(s)$: $Y(s)[s^2 + 2s + 2] - (s+3) = \frac{1}{s}\left(e^{-2\pi s} - e^{-4\pi s}\right)$ → heaviside

$\hookrightarrow Y(s) = \underbrace{\frac{s+3}{s^2+2s+2}}_{\substack{\text{easy!} \\ \text{sin \& cos.}}} + \underbrace{\frac{e^{-2\pi s} - e^{-4\pi s}}{s(s^2+2s+2)}}_{\text{hard!}} = F_1(s) + F_2(s)$ → deal w/ this

Ignore exp. for now, write

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs + C}{s^2+2s+2} = F_2(s)$$

Solving: $A = 1/2$
$B = -1/2$
$C = -1$

So, $F_2(s) = \underbrace{\left(e^{-2\pi s} - e^{-4\pi s}\right)}_{\text{what about these?}}\Bigg[\underbrace{\overbrace{\frac{1}{2}\cdot\frac{1}{s}}^{F_3(s)}}_{\text{easy!}} \overbrace{- \frac{1}{2}\cdot\frac{s}{s^2+2s+2} - 1\cdot\frac{1}{s^2+2s+2}}^{F_4(s)}\Bigg]$

sin & cosine.

Recall: $\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)}$
where $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

if we can invert these, we can use the rule to invert the whole thing

$F_3(s) = \frac{1}{2}\cdot\frac{1}{s}$, $\mathcal{L}^{-1}\left\{\frac{1}{2}\cdot\frac{1}{s}\right\} = \frac{1}{2} =: f_3(t)$

$F_4(s)$ complete the square: $s^2 + 2s + 2 = (s+1)^2 - 1 + 2$
$= (s+1)^2 + 1^2$

$F_4(s) = -\frac{1}{2}\cdot\frac{s}{(s+1)^2+1^2} - \frac{1}{(s+1)^2+1^2}$ ← sin & cosine.

$= -\frac{1}{2}\frac{(s+1)}{(s+1)^2+1^2} + \left(\frac{1}{2} - 1\right)\frac{1}{(s+1)^2+1^2}$

$= -\frac{1}{2}\frac{(s-(-1))}{(s-(-1))^2+1^2} - \frac{1}{2}\cdot\frac{1}{(s-(-1))^2+1^2}$

$= -\frac{1}{2}\cdot e^{-t}\cos(t) - \frac{1}{2}e^{-t}\sin(t) = f_4(t)$

$= -\frac{1}{2}e^{-t}[\cos(t) + \sin(t)] = f_4(t).$

$F_2(s) = \left(e^{-2\pi s} - e^{-4\pi s}\right)(F_3(s) + F_4(s)),$

$= e^{-2\pi s}\cdot(F_3(s) + F_4(s)) - e^{-4\pi s}(F_3(s) + F_4(s))$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$

$\mathcal{L}^{-1}\{F_3(s) + F_4(s)\} = f_3(t) + f_4(t)$
$= \frac{1}{2} - \frac{1}{2}e^{-t}[\cos(t) + \sin(t)]$

Then, $\mathcal{L}^{-1}\left\{e^{-2\pi s}(F_3(s) + F_4(s))\right\} = (f_3(t-2\pi) + f_4(t-2\pi))\cdot u(t-2\pi)$

$\boxed{= \left[\frac{1}{2} - \frac{1}{2}e^{-(t-2\pi)}\left(\cos(t-2\pi) + \sin(t-2\pi)\right)\right]u(t-2\pi)}$

$\mathcal{L}^{-1}\left\{-e^{-4\pi s}(F_3(s) + F_4(s))\right\} = (f_3(t-4\pi) + f_4(t-4\pi))\,u(t-4\pi)$

$\boxed{-\left[\frac{1}{2} - \frac{1}{2}e^{-(t-4\pi)}\left(\cos(t-4\pi) + \sin(t-4\pi)\right)\right]u(t-4\pi)}$

$$\frac{s+3}{s^2+2s+2} = \frac{s+3}{(s+1)^2+1^2} = \frac{s+1}{(s+1)^2+1^2} + 2\cdot\frac{1}{(s+1)^2+1^2}$$

So, $\mathcal{L}^{-1}\left\{\frac{s+3}{s^2+2s+2}\right\} = e^{-t}\cos(t) + 2e^{-t}\sin(t)$
$= \boxed{e^{-t}[\cos(t) + 2\sin(t)]},$

Thus: $\mathcal{L}^{-1}\{Y(s)\} = \boxed{\begin{array}{l} = \left[\frac{1}{2} - \frac{1}{2}e^{-(t-2\pi)}\left(\cos(t) + \sin(t)\right)\right]u_{2\pi}(t) \\ -\left[\frac{1}{2} - \frac{1}{2}e^{-(t-4\pi)}\left(\cos(t) + \sin(t)\right)\right]u_{4\pi}(t) \\ + e^{-t}[\cos(t) + 2\sin(t)] \quad = y(t) \end{array}}$

Convolution: $(f*g)(t) = \boxed{\int_0^t f(t-\tau)g(\tau)\,d\tau}$

~~$\mathcal{L}\{f\cdot g\} \neq F(s)G(s) \neq \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$~~.

BUT!

$\mathcal{L}\{f*g\}(s) = F(s)G(s)$ where
$\mathcal{L}\{f\} = F(s)$
$\mathcal{L}\{g\} = G(s)$

Find $\mathcal{L}\{y(t)\}$, where $y(t) = \int_0^t \sin(3(t-\tau))\cdot e^{-a(t-\tau)}\cdot \underbrace{1}_{g(\tau)}\,d\tau$

so, find $\mathcal{L}\{f(t)\}$ and
$\mathcal{L}\{g(t)\}$

$\begin{cases} f(t) = \sin(3t)e^{-at} \\ g(t) = 1 \end{cases}$

$\hookrightarrow \mathcal{L}\{\sin(3t)e^{-at}\} = \frac{3}{(s+a)^2+3^2}$

$\mathcal{L}\{1\} = \frac{1}{s},$

Therefore, $\mathcal{L}\{f*g\} = \mathcal{L}\left\{\int_0^t \sin(3(t-\tau))e^{-a(t-\tau)}\cdot 1\,d\tau\right\}$

$= \frac{3}{[(s+a)^2+3^2]}\cdot\frac{1}{s}$

$= F(s)\cdot G(s)$

$F(s) = \ln\left(\frac{s+4}{s-3}\right) = \ln(s+4) - \ln(s-3)$

$F^{(1)}(s) = \underbrace{\frac{1}{s+4}}_{e^{-4t}} - \underbrace{\frac{1}{s-3}}_{-e^{3t}}$

$F(s) \longrightarrow F'(s)$, $\mathcal{L}^{-1}\{F'(s)\} = e^{-4t} - e^{3t} = f(t)$

then $\boxed{\begin{array}{l} \mathcal{L}^{-1}\{F(s)\} = (-t)^n\cdot f(t) \\ = -t\left(e^{-4t} - e^{-3t}\right) \end{array}}$

$F(s) = \frac{2}{s^3}$

$G(s) = \frac{a}{(s^2-a^2)}$

$Y(s) = \frac{2a}{s^3(s^2-a^2)}$

$= \frac{2}{s^3}\cdot\frac{a}{(s^2-a^2)}$

$= F(s)\cdot G(s)$

$\mathcal{L}\{t^2\}$, $t^2 = f(t)$ $\qquad$ $\mathcal{L}\{\sinh(t)\}$, $\sinh(t) = g(t)$

So, $\mathcal{L}^{-1}\{F(s)G(s)\} = f*g$

$= \int_0^t f(t-\tau)g(\tau)\,d\tau$

$= \boxed{\int_0^t (t-\tau)^2\cdot\sinh(\tau)\,d\tau}$