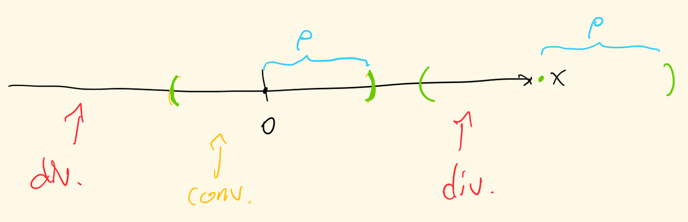
EL24 Lab 6 live

Thursday, March 4, 2021 3:57 PM

Power Series: $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ ← n → powers. (a_n ← coeff's, x_0 ← centre)



May not be valid everywhere!

There is a quantity ρ s.t. for $|x-x_0| < \rho$, it converges, $\rho \in [0, \infty)$.
otherwise it diverges.

(gives the interval.) (ρ: radius of convergence)

Can find ρ by considering:

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = L, \quad 0 \le L \le \infty.$$

(L is our radius of convergence.)

Ex. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and

$a_n = \frac{1}{n!}$

$(n+1)! = (n+1)\cdot n!$
$4! = 4\cdot 3\cdot 2\cdot 1$
$= 4 \cdot 3!$

So, $L = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right|$

$= \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right| = \lim_{n\to\infty}\left|(n+1)\cdot\frac{n!}{n!}\right|$

$= \lim_{n\to\infty}|n+1| = \infty.$

So, R.o.C. is $\rho = \infty$. So, series conv. on $(-\infty, \infty)$.

2.) $\sum_{n=0}^{\infty} (x-2)^n \cdot 1$ $\qquad a_n = 1.$

$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{1}\right| = 1.$

So, the R.o.C. is $\rho = 1$.
So, the series conv. for all x such that

↳ $|x-2| < 1.$
↳ $-1 < x-2 < 1$
$-1+2 < x-2+2 < 1+2$
↳ $1 < x < 3$ ⇝ I.o.C. $(1,3)$

Check ends:
$\sum_{n=0}^{\infty}(1-2)^n = \sum_{n=0}^{\infty}(-1)^n = -1 + (1) + (-1) + (1) + \cdots = -1$ $\quad = 0$

$\sum_{n=0}^{\infty}(3-2)^n = \sum_{n=0}^{\infty} 1 = 1+1+1+1+\cdots \;\infty$ Divergent!

Sol^n's to ODE's: Guess: $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ ← (solve for these!)

For "nice" power series (where it converges), we can do the "usual" stuff.

$y'(x) = \left(a_0 + a_1(x-x_0)^1 + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots\right)'$

$= 0 + a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots$

$= \sum_{n=1}^{\infty} a_n \cdot n (x-x_0)^{n-1}$

$y'' = \sum_{n=2}^{\infty} a_n \cdot n(n-1)(x-x_0)^{n-2}$

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1. $y' = cy$ about $x_0 = 0$

$y(x) = \sum_{n=0}^{\infty} a_n \cdot (x)^n$

$y'(x) = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1}$ $\qquad$ Plug it in!

$0 = y' - cy$

$= \sum_{n=1}^{\infty} a_n \cdot n x^{n-1} - c\sum_{n=0}^{\infty} a_n x^n$

send $n \mapsto n+1$

$= \sum_{n+1=1}^{\infty} a_{n+1}(n+1)x^{n+1-1} - c\sum_{n=0}^{\infty} a_n x^n$

$= \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - c\sum_{n=0}^{\infty} a_n x^n$

$0 = \sum_{n=0}^{\infty}\left[a_{n+1}(n+1) - ca_n\right]x^n$ ($=0$ for all n)

Recurrence Relation: $0 = a_{n+1}(n+1) - c\cdot a_n$ $\qquad a_n = \frac{(n+1)a_{n+1}}{c}$

or: $a_{n+1} = \frac{c\cdot a_n}{(n+1)}$

n=0: $a_1 = \frac{c^1 a_0}{1!}$

n=1: $a_2 = \frac{c\cdot a_1}{2} = \frac{c}{2}\cdot\frac{c}{1}\cdot a_0$

n=2: $a_3 = \frac{c\cdot a_2}{3} = \frac{c}{3}\cdot\frac{c}{2}\cdot\frac{c}{1}\cdot a_0$

n=3: $a_4 = \frac{c\cdot a_3}{4} = \frac{c}{4}\cdot\frac{c}{3}\cdot\frac{c}{2}\cdot\frac{c}{1}\cdot a_0$

n: $a_n = \frac{c^n}{n!}\cdot a_0$ ↔ $a_{n+1} = \frac{c^{n+1}}{(n+1)!}\cdot a_0$

$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_0 \cdot \frac{c^n}{n!} x^n$

$= a_0 \sum_{n=0}^{\infty} \frac{(cx)^n}{n!}$ $\qquad \sum_{n=0}^{\infty}\frac{(w)^n}{n!} = e^w$, $w = cx$, so

$= a_0 e^{cx}.$

Solve $y' + 2xy = 0$ about $x_0 = 0$

$y(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$ ←

$y'(x) = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$ ←

$0 = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1} + 2x \cdot \sum_{n=0}^{\infty} a_n x^n$

$= \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^{n+1}$

$n \mapsto n+1$

$= \sum_{n=0}^{\infty} a_{n+1}(n+1)\cdot x^n + \sum_{n=0}^{\infty} 2a_n x^{n+1}$

(starts with x^0 — has an extra term!) (starts at x^1)

$= a_1 + \sum_{n=1}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=0}^{\infty} 2a_n x^{n+1}$

$n \mapsto n+1$

$= a_1 + \sum_{n=0}^{\infty} a_{n+2}(n+2)x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1}$

$0 = a_1 x + \sum_{n=0}^{\infty}\left[a_{n+2}(n+2) + 2a_n\right]x^{n+1}$ (x^{n+1} not zero; bracket $=0$ for all n)

(Finding the recurrence relation)

$a_1 = 0$ $\qquad a_{n+2} = \frac{-2}{n+2}\cdot a_n$

n=0: $a_2 = \frac{-2}{2}\cdot a_0$ $\qquad$ n=1: $a_3 = \frac{-2}{3}\cdot a_1 = 0$

n=2: $a_4 = \frac{-2}{4}\cdot a_2$ $\qquad$ n=3: $a_5 = \frac{-2}{5}a_3 = 0$

$= \frac{-2}{4}\cdot\frac{-2}{2}\cdot a_0$

n=4: $a_6 = \frac{-2}{6}\cdot a_4$ $\qquad a_{2n+1} = 0$ → all odd coeff.'s are zero!

$= \frac{-2}{6}\cdot\frac{-2}{4}\cdot\frac{-2}{2}\cdot a_0$

$a_2 = -1\cdot a_0$ $\qquad a_4 = -\frac{1}{2}\cdot\frac{-1}{1}\cdot a_0$ $\qquad a_6 = -\frac{1}{3}\cdot\frac{-1}{2}\cdot\frac{-1}{1}\cdot a_0$

$a_{2\cdot4} = \frac{-1}{4}\cdot\frac{-1}{3}\cdot\frac{-1}{2}\cdot\frac{-1}{1}\cdot a_0$

$a_{2n} = \frac{(-1)^n}{n!}a_0$ $\qquad a_{2n+1} = 0$

$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_2x^2 + a_4x^4 + a_1x^1 + a_3x^3 + a_5x^5$ ($=0$)

$= a_0\cdot x^0 + a_2\cdot(x^2)^1 + a_4\cdot(x^2)^2 + a_6(x^2)^3 + a_8(x^2)^4 + \cdots$

$= a_{2\cdot0}(x^0) + a_{2\cdot1}(x^2) + a_{2\cdot2}(x^4) + a_{2\cdot3}(x^6) + a_{2\cdot4}(x^2)^4 + \cdots$

$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot a_0 (x)^{2n}$

$= a_0\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = a_0 e^{-x^2}$

$y' + 2xy = 0$

↳ $\frac{y'}{y} = -2x$

↳ $\int \frac{d}{dx}(\ln|y|) = \int -2x$

$\ln|y| = -x^2 + C$

↳ $y(x) = e^{-x^2}\cdot C$

$\cos(x) = \sum_{n=0}^{\infty}\frac{x^{2n}(-1)^n}{(2n)!}$