EL28 Lab 8 live

Thursday, March 18, 2021 9:39 AM

- IVP's w/ Laplace
- Convolutions

$$\begin{array}{ccc} \text{IVP} & \xrightarrow{\text{solve}} & \text{Sol}^{n} \\ \downarrow \text{1.} & & \uparrow \text{invert 3.} \\ \text{Laplace} & \xrightarrow[\text{2.}]{\text{work to find}} & Y(s) \end{array}$$

$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$

$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)$

What about: $\mathcal{L}\{u_c(t) f(t)\}$ $\quad t-c$

$$u(t-a) f(t) \to \int_0^\infty e^{-st} u(t-a) f(t)\,dt = \int_a^\infty e^{-st} f(t)\,dt = F(s) - \int_0^a e^{-st} f(t)\,dt$$

set $t = \tau + a$
$dt = d\tau$
$t = a \Rightarrow \tau = 0$
$t = \infty \Rightarrow \tau = \infty$

$$= \int_0^\infty e^{-s(\tau+a)} f(\tau + a)\,dt$$

Ex. $\begin{cases} y'' - 2y' + 5y = -8e^{-t} \\ y(0) = 2 \\ y'(0) = 12 \end{cases}$

1.
$\mathcal{L}\{y''\} = s^2 Y(s) - s y(0) - y'(0)$
$\mathcal{L}\{-2y'\} = -2(sY(s) - y(0))$
$\mathcal{L}\{y\} = Y(s)$
$\mathcal{L}\{-8e^{-t}\} = -8 \cdot \frac{1}{s+1}$

$$(s^2 Y(s) - s \cdot 2 - 12) - 2(sY(s) - 2) + 5Y(s) = \frac{-8}{s+1}$$

$Y(s) = \ ?$

$$(s^2 - 2s + 5) Y(s) = 2s + 12 - 4 - \frac{8}{s+1} = \frac{(2s+8)(s+1) - 8}{s+1} = \frac{2s^2 + 10s}{s+1}$$

2. $$\boxed{Y(s) = \frac{2s^2 + 10s}{(s+1)(s^2 - 2s + 5)}}$$

3a. Write in a familiar form! Partial fractions!

$$Y(s) = \frac{As + B}{s^2 - 2s + 5} + \frac{C}{s+1}$$

$$= \frac{(As+B)(s+1) + C(s^2 - 2s + 5)}{(s+1)(s^2 - 2s + 5)}$$

$$= \frac{s^2\overbrace{(A + C)}^{=2} + s\overbrace{(A + B - 2C)}^{=10} + \overbrace{(B + 5C)}^{=0}}{(s+1)(s^2 - 2s + 5)}$$

$\begin{cases} A + C = 2 & \Rightarrow C = 2 - A \\ A + B - 2C = 10 & \\ B + 5C = 0 & \Rightarrow B = -5C = -5(2 - A) = 5A - 10 \end{cases}$

$\to A + (5A - 10) - 2(2 - A) = 10$
$\to A(1 + 5 + 2) = 24$
$A = \frac{24}{8} = 3$
$\to C = 2 - 3 = -1$
$\to B = -5(-1) = 5$

$$\boxed{\begin{aligned} A &= 3 \\ B &= 5 \\ C &= -1 \end{aligned}}$$

$$\boxed{Y(s) = \frac{3s + 5}{s^2 - 2s + 5} - \frac{1}{s+1}}$$

some more work → complete the square

easy

$$\frac{3s + 5}{s^2 - 2s + 5} = \frac{3s + 5}{(s-1)^2 + 2^2}$$

$$= \frac{3(s-1) + 3 + 5}{(s-1)^2 + 2^2}$$

$$= 3 \cdot \frac{(s-1)}{(s-1)^2 + 2^2} + \frac{8}{(s-1)^2 + 2^2}$$

$$= \underbrace{3 \frac{(s-1)}{(s-1)^2 + 2^2}}_{\substack{\text{shifted cos!} \\ \text{shift is 1,} \\ b=2}} + \underbrace{4 \cdot \frac{2}{(s-1)^2 + 2^2}}_{\substack{\text{shifted sin!} \\ \text{shift is 1} \\ b=2}}$$

Take our inverse of each!

$\mathcal{L}^{-1}\left\{3 \frac{(s-1)}{(s-1)^2 + 2^2}\right\} = 3e^{t} \cdot \cos(2t)$

$\mathcal{L}^{-1}\left\{4 \cdot \frac{2}{(s-1)^2 + 2^2}\right\} = 4e^{t} \sin(2t)$

$\mathcal{L}^{-1}\left\{\frac{-1}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s - (-1)}\right\} = -e^{-t}$

So! $\mathcal{L}^{-1}\{Y(s)\} = \boxed{y(t) = 3e^{t}\cos(2t) + 4e^{t}\sin(2t) - e^{-t}}$

Ex. $\begin{cases} y'' + 2y' + 2y = u(t - 2\pi) - u(t - 4\pi) = (u_{2\pi}(t) - u_{4\pi}(t)) \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$

$$\boxed{u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}}$$

1. Laplace of both sides:

$$\mathcal{L}\{y'' + 2y' + 2y\} = s^2 Y(s) - s y(0) - y'(0) + 2(sY(s) - y(0)) + 2Y(s)$$
$$= Y(s)[s^2 + 2s + 2] - (s + 3)$$

Recall: $\mathcal{L}\{u(t-a)\} = e^{-as}/s$

So, $\mathcal{L}\{u(t - 2\pi) - u(t - 4\pi)\} = (\mathcal{L}\{u_{2\pi} - u_{4\pi}\}$

$$= \frac{e^{-2\pi s}}{s} - \frac{e^{-4\pi s}}{s}$$

2. Solve for $Y(s)$: $Y(s)[s^2 + 2s + 2] = (s + 3) + \frac{e^{-2\pi s} - e^{-4\pi s}}{s}$

$$\to Y(s) = \underbrace{\frac{s + 3}{s^2 + 2s + 2}}_{\text{easy}} + \underbrace{\frac{e^{-2\pi s}(1 - e^{-2\pi s})}{s(s^2 + 2s + 2)}}_{\text{work}}$$

3. Invert.

3a. More familiar! $\frac{1}{s(s^2 + 2s + 2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}$

$$\Rightarrow \boxed{A = \tfrac{1}{2},\ B = -\tfrac{1}{2},\ C = -1}$$

$$\frac{e^{-2\pi s}(1 - e^{-2\pi s})}{s(s^2 + 2s + 2)} = e^{-2\pi s}(1 - e^{-2\pi s})\left[\frac{1}{2} \cdot \frac{1}{s} + \left(-\frac{s}{2} - 1\right) \cdot \frac{1}{s^2 + 2s + 2}\right]$$

$$= \frac{e^{-2\pi s}(1 - e^{-2\pi s})}{2} \cdot \frac{1}{s} = F(s) - \frac{e^{-2\pi s}(1 - e^{-2\pi s})}{2} \cdot \frac{s}{s^2 + 2s + 2} = F(s)$$

$$- e^{-2\pi s}(1 - e^{-2\pi s}) \cdot \frac{1}{s^2 + 2s + 2} \to F(s)$$

Complete the square: $(s + 1)^2 + 1^2 = (s^2 + 2s + 2)$

Recall: $\boxed{\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a) \cdot u(t-a)}$

where $\mathcal{L}^{-1}\{F(s)\} = f(t)$,

So, we only need: $\mathcal{L}^{-1}\{\underbrace{\tfrac{1}{s}}_{F_1(s)}\}$, $\mathcal{L}^{-1}\{\underbrace{\tfrac{s}{(s+1)^2 + 1^2}}_{F_2(s)}\}$, $\mathcal{L}^{-1}\{\underbrace{\tfrac{1}{(s+1)^2 + 1^2}}_{F_3(s)}\}$

$\mathcal{L}^{-1}\{\frac{1}{2} \frac{1}{s}\} = \frac{1}{2} = f_1(t)$

$\frac{s}{(s+1)^2 + 1^2} = \frac{s + 1}{(s+1)^2 + 1^2} \boxed{- \frac{1}{(s+1)^2 + 1^2}}$

$\mathcal{L}^{-1}\left\{\frac{s + 1}{(s+1)^2 + 1^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s - (-1)}{(s - (-1))^2 + 1^2}\right\} = e^{-t}\cos(t) = f_2(t)$

$\mathcal{L}^{-1}\left\{-\frac{3}{2} \cdot \frac{1}{(s+1)^2 + 1^2}\right\} = -\frac{3}{2} e^{-t} \sin(t) = f_3(t)$

$\boxed{F_1(s) = \frac{1}{2s}}$, $\boxed{F_2(s) = \frac{s + 1}{(s+1)^2 + 1^2}}$, $F_3(s) = \boxed{-\frac{3}{2} \cdot \frac{1}{(s+1)^2 + 1^2}}$

So,

$\mathcal{L}^{-1}\{e^{-2\pi s} F_1(s)\} = f_1(t - 2\pi) u(t - 2\pi) = \frac{u(t - 2\pi)}{2}$

$\mathcal{L}^{-1}\{-e^{-4\pi s} F_1(s)\} = -\frac{u(t - 4\pi)}{2}$

$\mathcal{L}^{-1}\{e^{-2\pi s} F_2(s)\} = f_2(t - 2\pi) \cdot u(t - 2\pi) = e^{-(t - 2\pi)} \cdot \cos(t - 2\pi) \cdot u(t - 2\pi)$

$\mathcal{L}^{-1}\{-e^{-4\pi s} F_2(s)\} = e^{-(t - 4\pi)} \underbrace{\cos(t - 4\pi)}_{= \cos(t)} u(t - 4\pi)$

$\mathcal{L}^{-1}\{e^{-2\pi s} F_3(s)\} = -\frac{3}{2} e^{-(t - 2\pi)} \cdot \underbrace{\sin(t - 2\pi)}_{= \sin(t)} u(t - 2\pi)$

$\mathcal{L}^{-1}\{e^{-4\pi s} F_3(s)\} =$

$$\mathcal{L}^{-1}\left\{e^{-2\pi s}[F_1(s) + F_2(s) + F_3(s)]\right\}$$
$$= u(t - 2\pi)[f_1(t - 2\pi) + f_2(t - 2\pi) + f_3(t - 2\pi)]$$

Finally, $\mathcal{L}^{-1}\{Y(s)\} = y(t) = e^{-t}[\sin(t) + 2\cos(t)]$
$$+ u(t - 2\pi)\left[\frac{1}{2} - \frac{3}{2} e^{-(t - 2\pi)} \sin(t) + e^{-(t - 2\pi)} \cos(t)\right]$$
$$+ u(t - 4\pi)\left[\frac{1}{2} - \frac{3}{2} e^{-(t - 4\pi)} \sin(t) + e^{-(t - 4\pi)} \cos(t)\right]$$

$(f * g)(t) = \int_0^t f(t - \tau) g(\tau)\, d\tau$

$\to \mathcal{L}\{f * g\}(s) = F(s)\, G(s)$ where

$\mathcal{L}\{f\} = F(s)$
$\mathcal{L}\{g\} = G(s)$

$\to f * g = \mathcal{L}^{-1}\{F(s) G(s)\}$.

$\cancel{\mathcal{L}\{f \cdot g\} \neq F(s) G(s)}$

$\mathcal{L}\{f * g\} = F(s) G(s)$

$f(t) = \sin(t) \longrightarrow F(s) = \frac{a}{s^2 + a^2}$

$g(t) = e^{at} \longrightarrow G(s) = \frac{1}{s - a}$

Given: $\int_0^t \underbrace{\sin(t - \tau)}_{f(t-\tau)} \cdot \underbrace{e^{a\tau}}_{g(\tau)}\, d\tau$

So, $\mathcal{L}\left\{\int_0^t \sin(t - \tau) e^{a\tau} d\tau\right\} = F(s) \cdot G(s) = \frac{1}{(s^2 + 1)} \cdot \frac{1}{(s - a)}$

where $F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s^2 + 1}$

$G(s) = \mathcal{L}\{g(t)\} = \frac{1}{s - a}$