EL21 Lab 6 live

Thursday, March 4, 2021 9:43 AM

Ex. $\boxed{y' = cy}$. solve this about $x_0 = 0$

1. Assume that $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

coeff's centre

So, $y'(x) = \sum_{n=1}^{\infty} a_n n \cdot x^{n-1}$

2. Plug into eq^{n}: $y' - ay = 0$

$\hookrightarrow \sum_{n=1}^{\infty} a_n n x^{n-1} - c \sum_{n=0}^{\infty} a_n x^n = 0$

$n \mapsto n+1$

3. Match powers of x!

$\sum_{n+1=1}^{\infty} a_{n+1}(n+1)\, x^{(n+1)-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)\, x^n$

$\rightsquigarrow$ set $n = m+1$ then relabel $m \mapsto n$.

So, $\sum_{n=0}^{\infty} \underbrace{[a_{n+1}(n+1) - c\, a_n]}_{=0 \text{ for every } n} x^n = 0$

recurrence relation!

$\hookrightarrow a_{n+1}(n+1) - c \cdot a_n = 0$

$\hookrightarrow \boxed{a_{n+1} = \frac{c}{n+1} \cdot a_n}$.

4. Solve the recurrence.

$n=0$: $a_1 = \frac{c}{1} \cdot a_0$

$n=1$: $a_2 = \frac{c}{2} \cdot a_1 = \frac{c}{2} \cdot \frac{c}{1} a_0$

$n=2$: $a_3 = \frac{c}{3} \cdot a_2 = \frac{c}{3} \cdot \frac{c}{2} \cdot \frac{c}{1} \cdot a_0$

$n=3$: $a_4 = \frac{c}{4} \cdot a_3 = \frac{c}{4} \cdot \frac{c^3}{3 \cdot 2 \cdot 1} = \frac{c^4}{4!}$

$n+1$

$\boxed{a_n = \frac{1}{n!} \cdot c^n \cdot a_0}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Plug in to original sol^{n}: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$= \sum_{n=0}^{\infty} a_0 \cdot \frac{c^n \cdot x^n}{n!}$

$= a_0 \sum_{n=0}^{\infty} \frac{(cx)^n}{n!}$

$y = e^{cx}$

solves $y' = cy$.

Ex. $y' + 2xy = 0$ about $x_0 = 0$.

$y(x) = \sum_{n=0}^{\infty} a_n x^n$

$y'(x) = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1}$

$y' + 2xy = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n$

$= \sum_{n=1}^{\infty} a_n \cdot n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^{n+1}$

reindex. send $n \mapsto n+1$.

leave above for now

$= \sum_{n+1=1}^{\infty} a_{n+1}(n+1)\, x^{(n+1)-1} + \sum_{n=0}^{\infty} 2a_n x^{n+1}$

$\star = \sum_{n=0}^{\infty} a_{n+1}(n+1)\, x^n + \sum_{n=0}^{\infty} 2a_n x^{n+1}$

starts with a constant

starts at x^1

$= a_1 \cdot (1) \cdot x^0 + \sum_{n=1}^{\infty} a_{n+1}(n+1)\, x^n + \sum_{n=0}^{\infty} 2a_n x^{n+1}$

$(n+1)a_{n+1} + 2a_{n+1}$

$= (n+3)a_{n+1}$

$\Rightarrow a_{n+1} = 0$

$= a_1 + \sum_{n=1}^{\infty} a_{n+1}(n+1)\, x^n + \sum_{n=0}^{\infty} 2a_n x^{n+1}$

$n \mapsto n+1$

$\star = a_1 + \sum_{\substack{n+1=1 \\ n=0}}^{\infty} a_{n+2}(n+2)\, x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1}$

$0 = a_1 + \sum_{n=0}^{\infty} \underbrace{[a_{n+2}(n+2) + 2a_n]}_{=0} x^{n+1}$

starts at x^1.
(not constant!!)

$\boxed{a_1 = 0}$

$a_{n+2}(n+2) + 2a_n = 0$

$\hookrightarrow a_{n+2}(n+2) = -2a_n$

$\hookrightarrow \boxed{a_{n+2} = \frac{-2}{(n+2)} \cdot a_n}$ and $\boxed{a_1 = 0}$

sep. by 2.
$\hookrightarrow$ even & odd cases

Even:

$n=0$: $a_2 = -\frac{2}{2} \cdot a_0$

$n=2$: $a_4 = -\frac{2}{4} \cdot a_2$

$= -\frac{2}{4} \cdot -\frac{2}{2} \cdot a_0$

$n=4$: $a_6 = -\frac{2}{6} \cdot a_4$

$= \left(-\frac{2}{6} \cdot -\frac{2}{4} \cdot -\frac{2}{2} \cdot a_0\right) = -\frac{1}{3} \cdot -\frac{1}{2} \cdot -\frac{1}{1}$

$n=6$: $a_8 = -\frac{2}{8} \cdot a_6$

$= -\frac{2}{8} \cdot -\frac{2}{6} \cdot -\frac{2}{4} \cdot -\frac{2}{2} \cdot a_0$

$= -\frac{1}{4} \cdot -\frac{1}{3} \cdot -\frac{1}{2} \cdot -\frac{1}{1} \cdot a_0$

Odd:

$n=1$: $a_3 = -\frac{2}{3} \cdot a_1 = 0$

$n=3$: $a_5 = -\frac{2}{5} \cdot a_3 = 0$

$a_{2n+1} = 0$

$a_{2n} = \frac{(-1)^n}{n!} \cdot a_0 \qquad a_{2n+1} = 0$

$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$

$= a_0 \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = a_0 e^{-x^2}$

$y' + 2xy = 0$

$\hookrightarrow \frac{y'}{y} = -2x$

$\frac{d}{dx}\left(\ln(y)\right) = -2x$

$\hookrightarrow \ln(y) = -x^2 + C$

$y = C e^{-x^2}$