EL24 Lab 4 live

Thursday, February 11, 2021 3:59 PM

- one "tough" U.C.'s problem
- Variation of parameters

Ex.

$$y'' + 2y' + 5y = 4e^{-t}\cos(2t)$$

$tAe^{-t}\cos(2t) + tBe^{-t}\sin(2t)$

$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

1. Hom. Sol^{n}:

$r^2 + 2r + 5 = 0$

$\hookrightarrow \frac{-2 \pm \sqrt{4-4\cdot1\cdot5}}{2\cdot1} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

$$y_h(t) = e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t))$$
$$= e^{-t}(c_1\cos(2t) + c_2\sin(2t))$$

2. Need to make a good guess → Right hand side is similar to hom. sol^{n}!

1. $y_p = Ae^{-t}\cos(2t)$ (crossed out)
2. $y_p = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)$ (crossed out) → same as hom. sol^{n}!
3. $y_p = Ate^{-t}\cos(2t)$ → not enough information!! plug in and find $A = 0$.
4. $y_p = Ate^{-t}\cos(2t) + Bte^{-t}\sin(2t)$ — throw away
5. $y_p = (A_1t^2 + A_2t + A_3)e^{-t}\cos(2t) + (B_1t^2 + B_2t + B_3)e^{-t}\sin(2t)$

too much information! Find $A_1 = B_1 = 0$

Key steps:

$$y_p(t) = Ate^{-t}\cos(2t) + Bte^{-t}\sin(2t)$$
$$= te^{-t}(A\cos(2t) + B\sin(2t))$$

$$y_p'(t) = e^{-t}(A\cos(2t) + B\sin(2t)) + (-1)y_p(t) + te^{-t}(-2A\sin(2t) + 2B\cos(2t))$$

$$y_p''(t) = -e^{-t}(A\cos(2t) + B\sin(2t)) + e^{-t}(-2A\sin(2t) + 2B\cos(2t)) \; ☆$$
$$- y_p'(t)$$
$$+ e^{-t}(-2A\sin(2t) + 2B\cos(2t)) \; ☆$$
$$- te^{-t}(-2A\sin(2t) + 2B\cos(2t)) \; ☆$$
$$+ te^{-t}(-4A\cos(2t) - 4B\sin(2t)) \; ☆$$

Eq^{n}: $y_p'' + 2y_p' + 5y_p = ☆ + ☆ + ☆ + ☆ - y_p'(t) + 2y_p' + 5y_p$

$= ☆\text{'s} + y_p' + 5y_p$

$= ☆\text{'s} + ☆\text{'s} - y_p(t) + 5y_p$

$= ☆\text{'s} + ☆\text{'s} + 4y_p$

$= 4e^{-t}\cos(2t)$

Match coeff's of: $e^{-t}\cos(2t), e^{-t}\sin(2t), te^{-t}\cos(2t), te^{-t}\sin(2t)$ → $A = 0, \; B = 1$

$y_p = 1 \cdot t \cdot e^{-t} \cdot \sin(2t)$

Okay, full sol^{n}: $y = y_h + y_p$

$$= e^{-t}(c_1\cos(2t) + c_2\sin(2t)) + te^{-t}\sin(2t)$$

Variation of Parameters: $y'' + P(x)y' + Q(x)y = R(x)$

$y_p(x) = y_1u_1 + y_2u_2$, where y_1, y_2 are from hom. sol^{n}, and

$$u_1 = -\int \frac{y_2R(x)}{W(y_1,y_2)}dx$$
$$u_2 = +\int \frac{y_1R(x)}{W(y_1,y_2)}dx$$

$$y_p = y_1u_1 + y_2u_2$$

Ex. $y'' + 2y' + y = e^{-x}\sin(x)$.

1. hom. sol^{n}: $r^2 + 2r + 1 = 0$

$\hookrightarrow \frac{-2 \pm \sqrt{4-4\cdot1\cdot1}}{2\cdot1} = -1$ real, repeated.

$y_h = c_1y_1 + c_2y_2 = c_1e^{-x} + c_2xe^{-x}$ ($y_1 = e^{-x}$, $y_2 = xe^{-x}$)

2. Wronskian!

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}$$
$$= e^{-x}(e^{-x} - xe^{-x}) + xe^{-x}e^{-x}$$
$$= e^{-2x} \neq 0 \quad \text{(good news)}$$

3. Solve u_1 & u_2:

$$u_1 = -\int \frac{y_2R}{W(y_1,y_2)}dx = -\int \frac{xe^{-x}e^{-x}\sin(x)}{e^{-2x}} = -\int x\sin(x)\,dx$$

$u = x$, $dv = \sin(x)$; $du = 1$, $v = \cos(x)$

$$= x\cos(x) - \int\cos(x) = x\cos(x) - \sin(x)$$

$$u_2 = +\int \frac{y_1R(x)}{W(y_1,y_2)}dx = \int \frac{e^{-x}e^{-x}\sin(x)}{e^{-2x}} = \int\sin(x) = -\cos(x)$$

4. Put it together!

$$y_p(x) = y_1u_1 + y_2u_2$$
$$= e^{-x}(x\cos(x) - \sin(x)) + xe^{-x}(-\cos(x))$$
$$= -e^{-x}\sin(x)$$

So, $y = y_h + y_p$
$$= c_1e^{-x} + c_2xe^{-x} - e^{-x}\sin(x)$$

Ex. $y'' + y = \tan(x)$ V.o.P.!

1. hom. sol^{n}: $r^2 + 1 = 0$

$\hookrightarrow r = \pm i$

So, $c_1y_1(x) = c_1\cos(x)$ → y_1

$c_2y_2(x) = c_2\sin(x)$ → y_2

2. Wronskian!

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$
$$= \cos^2(x) + \sin^2(x)$$
$$= 1 \quad \text{(nice!)}$$

3. Solve u_1 & u_2:

$$u_1 = -\int \frac{y_2R(x)}{W(y_1,y_2)}dx = -\int \frac{\sin(x)\tan(x)}{1} = -\int \sin(x)\cdot\frac{\sin(x)}{\cos(x)}$$
$$= -\int \frac{\sin^2(x)}{\cos(x)}dx \quad \to \text{use } \sin^2(x) = 1 - \cos^2(x)$$

$-\sin^2(x) = -(1-\cos^2(x)) = \cos^2(x) - 1$

$$= \int \frac{\cos^2(x) - 1}{\cos(x)} = \int \cos(x) - \sec(x)$$
$$= \sin(x) - \ln|\sec(x) + \tan(x)|$$

$$u_2 = +\int \frac{y_1R(x)}{W(y_1,y_2)}dx = \int \frac{\cos(x)\tan(x)}{1} = \int \cos(x)\frac{\sin(x)}{\cos(x)}$$
$$= \int\sin(x) = -\cos(x)$$

4. Put it together!

$$y_p = y_1u_1 + y_2u_2$$
$$= \cos(x)(\sin(x) - \ln|\sec(x) + \tan(x)|) + \sin(x)(-\cos(x))$$
$$= -\cos(x)\ln|\sec(x) + \tan(x)|$$

Final sol^{n}:

$$y = y_h + y_p$$
$$= c_1\cos(x) + c_2\sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|$$