EL24 Lab 3 live

Thursday, February 4, 2021 3:57 PM

- Wronskian
- Undetermined coefficients

Ex. Compute the wronskian of the solution set (on an interval including the origin)
$y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$.
Show when they form a fund. sol^{n} set.

$$W[y_1, y_2](t_0) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= (y_1 y_2' - y_2 y_1')\Big|_{t_0=0}$$

$$= \left(e^{r_1 t_0} r_2 e^{r_2 t_0} - e^{r_2 t_0} r_1 e^{r_1 t_0}\right)\Big|_{t_0=0}$$

$$= (r_2 - r_1) e^{(r_1 + r_2) t_0}\Big|_{t_0=0}$$

$$= r_2 - r_1 = W(y_1, y_2)(0)$$

$y_1' = r_1 e^{r_1 t}$
$y_2' = r_2 e^{r_2 t}$

$$\begin{cases} y'' + y' + y = 0, & t > 1 \\ y(1) = 1 \\ y'(1) = 2 \end{cases}$$
$t_0 = 1$

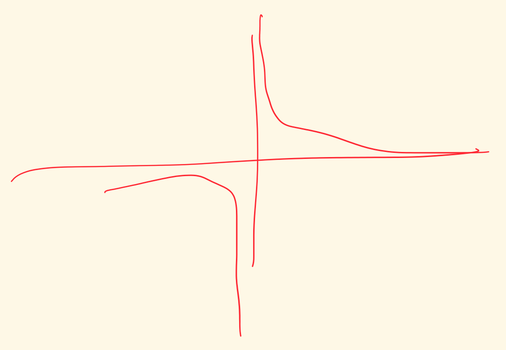
(1) when $r_1 = r_2$: not linearly indep or not a fund sol^{n} set

(2) $r_1 \neq r_2$, we have a fund. sol^{n} set.

Ex. Can $y_1(t) = \sin(t^2)$ be a piece of a fund. sol^{n}. set on an interval containing the origin of a second order hom. eq^{n} with continuous coefficients? → sol^{n}'s are continuous

Wronskian take $t_0 = 0$.
Suppose $y_2(t)$ is any other solution.

$y_1' = \cos(t^2) \cdot 2t$
$= 2t\cos(t^2)$



$$W[y_1, y_2](t_0) = \begin{vmatrix} \sin(t_0^2) & y_2(t_0) \\ 2t\cos(t_0^2) & y_2'(t_0) \end{vmatrix}$$

$$= \left[\sin(t_0^2) \cdot y_2'(t_0) - 2t_0 \cos(t_0^2) y_2(t_0)\right]\Big|_{t_0=0}$$

$$= 0 \cdot y_2'(t_0) - 0 \cdot 1 \cdot y_2(t_0)$$

$$= 0, \quad \neq 0$$

⇒ NO! y_1 cannot be piece of a fund. sol^{n} set containing the origin.

what if $y_2 = \frac{1}{t}$?

$W = 0 - 2t \cdot \frac{1}{t}$
$= -2 < 0$

Undetermined Coefficients: Solve problems like:

$$ay'' + by' + cy = Q(x) \rightarrow y_p$$

Main idea: Guess a solution based on the "form" of Q and solve for some coefficients.

Ex. $y'' + y' - 2y = x^2 = Q(x)$.

1. Find the homogeneous sol^{n}!

$r^2 + r - 2 = 0 \Rightarrow \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-2)}}{2} = -\frac{1}{2} \pm \frac{3}{2}$ → 1, -2

So, $y_h(x) = c_1 e^{x} + c_2 e^{-2x}$

2. Find the particular solution through a "good" guess:

$y_p(x) = Ax^2 + Bx + C$, and plug in & solve for A, B, C.

$x^4 \rightarrow Ax^4 + Bx^3 + Cx^2 + Dx + E$.

$y = Ax^2$
$y = Ax^2 + C$

$Ax^2 + Bx + C$ ($B \to 0$)

$y_p'(x) = 2Ax + B$
$y_p''(x) = 2A$.

From the original eq^{n}:
$$x^2 = 1 \cdot (2A) + 1 \cdot (2Ax + B) - 2(Ax^2 + Bx + C)$$
(y_p'', y_p', y_p)

$$1 \cdot x^2 + 0 \cdot x + 0 = (-2A)x^2 + (2A - 2B)x + (2A + B - 2C)$$

$$\begin{cases} -2A = 1 & \Rightarrow A = -1/2 \\ 2A - 2B = 0 & \Rightarrow B = A = -1/2 \\ 2A + B - 2C = 0 & \Rightarrow C = A + B/2 = -\frac{1}{2} + \frac{1}{2}\left(-\frac{1}{2}\right) = -\frac{3}{4} \end{cases}$$

So, $y_p(x) = -\frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}$

Finally, put it all together!

$$y(x) = y_h(x) + y_p(x)$$
$$y(x) = c_1 e^{x} + c_2 e^{-2x} + \left(-\frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}\right)$$

Ex. Solve $2y'' + 4y' + 2y = e^{-x}$.

1. Hom. sol^{n}!

$2r^2 + 4r + 2 = 0$
$\frac{-4 \pm \sqrt{16 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} = -1 \rightarrow$ one repeated root.

$y_h(x) = c_1 e^{-x} + c_2 x e^{-x}$

2. Solve $y_p(x)$ with a good guess.

~~Try: $y_p(x) = Ae^{-x}$~~ ✗ ⟶ $2y_p'' + 4y_p' + 2y_p = 0 \neq e^{-x}$

Try: $y_p(x) = (Ax + B)e^{-x}$ ✗
$= Axe^{-x} + Be^{-x}$

Try: $y_p(x) = (Ax^2 + Bx + C)e^{-x}$. ⟶
$= Ax^2e^{-x} + Bxe^{-x} + Ce^{-x}$ (hom. sol^{n}'s)

$c_1 e^{-x} + c_2 x e^{-x}$
$y_p(x) = \frac{x^2}{4}e^{-x} + Bxe^{-x} + Ce^{-x}$

$B = C = 0$

$$y_h + y_p = \overbrace{(c_1 + C)}^{d_1} e^{-x} + \overbrace{(c_2 + B)}^{d_2} x e^{-x} + \frac{1}{4}x^2 e^{-x}$$

$y_p(x) = Ax^2e^{-x}$
$y_p'(x) = 2Axe^{-x} - Ax^2e^{-x}$
$y_p''(x) = 2Ae^{-x} - 2Axe^{-x} - 2Axe^{-x} + Ax^2e^{-x}$
$= 2Ae^{-x} - 4Axe^{-x} + Ax^2e^{-x}$.

Plug into orig. eq^{n}:
$$e^{-x} = 2(2Ae^{-x} - 4Axe^{-x} + Ax^2e^{-x}) + 4(2Axe^{-x} - Ax^2e^{-x}) + 2(Ax^2e^{-x})$$
$$= e^{-x}(4A) + xe^{-x}(\underbrace{-8A + 8A}_{=0}) + x^2e^{-x}(\underbrace{2A - 4A + 2A}_{=0})$$

$1 \cdot e^{-x} = e^{-x} \cdot 4A$
⇒ $1 = 4A$
⇒ $A = 1/4$

So, $y_p(x) = \frac{x^2}{4}e^{-x} = Ax^2e^{-x}$

Putting it all together:
$$y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 x e^{-x} + \frac{x^2}{4}e^{-x}$$

$(y_h + y_p)' = y_h' + y_p'$
$y(x) = y_h + y_p$

$2y'' + 4y' + 2y = e^{-x}$
$2(y_h + y_p)'' + 4(y_h + y_p)' + 2(y_h + y_p)$
$= 2y_h'' + 4y_h' + 2y_h$ ($= 0$)
$+ 2y_p'' + 4y_p' + 2y_p$ ($= e^{-x}$)
$= 0 + e^{-x}$

$$\begin{cases} 2y'' + 4y' + 2y = 0 \\ y(0) = y'(0) = 7 \end{cases}$$

$y_p = \frac{1}{4}x^2e^{-x}$
$y_p(0) = 0$
$y_p'(0) = 0$
$y_h(x) = c_1 e^{-x} + c_2 x e^{-x}$

$c_1 = c_2 = 0$
$y(x) = y_p(x)$