EL24 Lab 2 live

Thursday, January 28, 2021 3:57 PM

Ex.

$$t^2 \frac{dy}{dt} + 2ty = y^3$$

1. Standard form!

$$\frac{dy}{dt} + \underbrace{\frac{2}{t}}_{P(t)} y = \underbrace{\frac{1}{t^2}}_{Q(t)} y^{\boxed{3}} \quad = n$$

$$\frac{1}{y^5} = y^{\boxed{-5}}$$

2. Convert to a linear eq^n via: $\boxed{u = y^{1-n}}$

Then solve: $\boxed{\frac{du}{dt} + (1-n)P(t)u = (1-n)Q(t)}$

$$\frac{dy}{dt} + P(t)y = Q(t)y^n$$

$$\frac{dy}{dt} = Q(t)y^n - P(t)y$$

$$u = y^{1-n}$$

$$\frac{du}{dt} = \frac{d}{dt}\left(y^{1-n}\right)$$

$$= (1-n)y^{-n} \cdot \frac{dy}{dt}$$

$$= (1-n)y^{-n}\left[Q(t)y^n - P(t)y\right]$$

$$= (1-n)Q(t) - (1-n)P(t)y^{1-n}$$

$$= (1-n)Q(t) - (1-n)P(t)u$$

$$\Rightarrow \boxed{\frac{du}{dt} + (1-n)P(t)u = (1-n)Q(t)}$$

So:

$$\frac{du}{dt} + (1-3)\frac{2}{t}u = (1-3)\frac{1}{t^2}$$

$$\boxed{\frac{du}{dt} - \underbrace{\frac{4}{t}}_{P(t)}u = -\underbrace{\frac{2}{t^2}}_{Q(t)}}$$

3. Solve for u using the int. fact. method.

$$\Rightarrow \mu(t) = e^{\int p(t)} = e^{-\int \frac{4}{t}dt} = e^{-4\ln(t)}$$

$$= e^{\ln(t^{-4})} = t^{-4}$$

$$u(t) = \frac{1}{\mu(t)}\int \mu(t)p(t)$$

Multiply eq^n by $\mu(t) = t^{-4}$:

$$\underbrace{t^{-4}\frac{du}{dt} - 4t^{-5}u}_{} = -2t^{-6}$$

$$\frac{d}{dt}\left(t^{-4}u(t)\right) = -4t^{-5}u(t) + t^{-4}u'(t)$$

$$\int \frac{d}{dt}\left(t^{-4}u\right) = \int -2t^{-6}$$

$$t^{-4}u = -2\left(-\frac{1}{5}t^{-5}\right) + C$$

$$= \frac{2}{5}t^{-5} + C$$

$$\Rightarrow \boxed{u(t) = \frac{2}{5}t^{-1} + Ct^4}$$

$\rightarrow$ remember: $u = y^{1-n}$

$$\Rightarrow y = u^{\frac{1}{1-n}} = u^{-\frac{1}{2}}$$

4. Convert back to find $y(t)$!

$$y(t) = u^{-1/2}$$

$$= \left(\frac{2}{5}t^{-1} + Ct^4\right)^{-1/2}$$

$$\boxed{y(t) = \frac{1}{\sqrt{\frac{2}{5}t^{-1} + Ct^4}}}$$

Second Order, linear, homogeneous, const. coeff. eq^n:

$$ay'' + by' + cy = 0$$

Guess: $\boxed{y(x) = e^{rx}}$

Then, $ay'' + by' + cy = \underbrace{e^{rx}}_{\neq 0}\underbrace{(ar^2 + br + c)}_{=0} = 0$

Characteristic Equation!

Roots: (call them r_1, r_2) $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\rightarrow$ discriminant

① $b^2 - 4ac > 0 \Rightarrow$ two real, distinct roots. $r_1, r_2 \in \mathbb{R}$

② $b^2 - 4ac = 0 \Rightarrow$ one real, repeated root, $r_1 = r_2 \in \mathbb{R}$

③ $b^2 - 4ac < 0 \Rightarrow$ complex conjugates! $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$. $(\alpha, \beta \in \mathbb{R})$

①: $2y'' + 4y' - y = 0$ $(a = 2, b = 4, c = -1)$

Char. Eq^n: $2r^2 + 4r - 1 = 0$

Roots: $\frac{-4 \pm \sqrt{16 - 4(2)(-1)}}{2(2)} = -1 \pm \frac{\sqrt{24}}{4}$

$$= -1 \pm \frac{\sqrt{6}}{2}$$

$$\begin{cases} r_1 = -1 + \sqrt{6}/2 \\ r_2 = -1 - \sqrt{6}/2 \end{cases}$$

In case I, the solution is:

$$\boxed{y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

$$= c_1 e^{(-1+\frac{\sqrt{6}}{2})x} + c_2 e^{(-1-\frac{\sqrt{6}}{2})x}$$

Ex. Solve $2y'' + 4y' + 2y = 0$ $(a = 2, b = 4, c = 2)$

Char. Eq^n: $2r^2 + 4r + 2 = 0$

Roots: $\frac{-4 \pm \sqrt{16 - 4(2)(2)}}{2(2)} = -1$

Case II: One real repeated root $r_1 = r_2 = -1$.

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$$= c_1 e^{-x} + c_2 e^{-x} = (c_1 + c_2)e^{-x}$$

$$= Ce^{-x}.$$

In order to find a second solution, multiply by x!

$$\boxed{y(x) = \underbrace{c_1 e^{-x}}_{1^{st}\text{ piece}} + \underbrace{c_2 x e^{-x}}_{2^{nd}\text{ piece}}}$$

Ex. Solve $\frac{1}{2}y'' + y' + 5y = 0$

Char. Eq^n: $\frac{1}{2}r^2 + r + 5 = 0$

Roots: $\frac{-1 \pm \sqrt{1 - 4(\frac{1}{2})(5)}}{2(\frac{1}{2})} = -1 \pm \sqrt{-9}$

$$= \underbrace{-1 \pm i3}_{\text{complex conjugates!}}$$

$\alpha = -1$, $\beta = 3$;

$$\begin{cases} r_1 = -1 + 3i \\ r_2 = -1 - 3i \end{cases}$$

In case III, solutions of the form:

$$\boxed{y(x) = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right) = e^{-x}\left[c_1\cos(3x) + c_2\sin(3x)\right]}$$

$$e^{ix} = \cos(x) + i\sin(x)$$

$$e^{r_1 x} = e^{(\alpha + i\beta)x} = e^{\alpha x} \cdot e^{i\beta x} = e^{\alpha x}\left[\cos(\beta x) + i\sin(\beta x)\right]$$

Case I: $r_1, r_2 \in \mathbb{R}$ $(r_1 \neq r_2)$ $\Rightarrow$ $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case II: $r_1 = r_2 \in \mathbb{R}$ $\Rightarrow$ $y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

Case III: $r_1, r_2 \in \mathbb{C}$ $\begin{pmatrix} r_1 = \alpha + i\beta \\ r_2 = \alpha - i\beta \end{pmatrix}$ $\Rightarrow$ $y(x) = e^{\alpha x}\left[c_1\cos(\beta x) + c_2\sin(\beta x)\right]$

Wronskian: Given two solutions $y_1(x), y_2(x)$, the Jacobian matrix is:

$$J(y_1, y_2) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$|A| = \det(A) = ad - bc$$

The Wronskian is the determinant of the Jacobian

$$\det(J) = \boxed{y_1 y_2' - y_1' y_2 \equiv W(y_1, y_2)}$$