EL21 Lab 4 live

Thursday, February 11, 2021 9:53 AM

- one "tough" undet. coeff's
- variation of parameters

Ex. $y'' + 2y' + 5y = \boxed{4e^{-t}\cos(2t)}$

$y(0) = 1, \quad y'(0) = 0$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$\to r^2 + 2r + 5$

$\to \dfrac{-2 \pm \sqrt{4-4\cdot 1\cdot 5}}{2} = \dfrac{-2 \pm i\sqrt{16}}{2} = -1 \pm i\dfrac{4}{2}$

$= \boxed{-1 \pm 2i}$

So the roots are $r_1 = -1 + 2i$

$r_2 = -1 - 2i$

Hom. Sol^{n}: of the form $e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t))$

$y_h(t) = e^{-t}(c_1\cos(2t) + c_2\sin(2t))$ → no t!

Yes!

Particular: need a good guess. $\cancel{Ae^{-t}\cos(2t)} + \cancel{Be^{-t}\sin(2t)}$

$\cancel{Ae^{-t}\cos(2t)}$ → hom sol^{n}!!

$Ate^{-t}\cos(2t)$ ⇝ not enough info!

plug in and find $A = 0$ ….

no good!

$$y_p(t) = (\cancel{A_1t^2} + A_2t + \cancel{A_3})e^{-t}\cos(2t)$$
$$(\cancel{B_1t^2} + B_2t + \cancel{B_3})e^{-t}\sin(2t)$$

too much information! Plug in and find $A_1 = B_1 = 0$

hom. sol^{n}

So! $y_p(t) = Ate^{-t}\cos(2t) + Bte^{-t}\sin(2t)$

Plug into eq^{n} to find: $A = 0, \quad B = 1$

So,

$$y_p(t) = te^{-t}\sin(2t)$$

$y(t) = y_h(t) + y_p(t) = e^{-t}(c_1\cos(2t) + c_2\sin(2t)) + te^{-t}\sin(2t)$

$y(0) = e^{0}(c_1\cos(0) + c_2\sin(0)) + 0$

$= c_1(1 + 0) = c_1$

$y(0) = 1 = c_1 \implies \boxed{c_1 = 1}$

$y'(t) = -e^{-t}(\cos(2t) + c_2\sin(2t)) + e^{-t}(-2\sin(2t) + 2c_2\cos(2t))$

$+ e^{-t}\sin(2t) - te^{-t}\sin(2t) + 2te^{-t}\cos(2t)$

$y'(0) = -e^{0}(\cos(0) + 0) + e^{0}(0 + 2c_2\cos(0))$

$+ 0 - 0 + 0$

$= -1 + 2c_2 = 0$

$\implies 2c_2 = 1$

$\implies \boxed{c_2 = 1/2}$

Final solution:
$$\boxed{y(t) = e^{-t}\left(\cos(2t) + \tfrac{1}{2}\sin(2t)\right) + te^{-t}\sin(2t)}$$

Variation of Parameters: Solve $y'' + P(x)y' + Q(x)y = R(x)$

$y_p = y_1u_1 + y_2u_2$, where

$$u_1 = -\int \frac{y_2R}{W(y_1,y_2)}\,dx,$$

$$u_2 = \int \frac{y_1R}{W(y_1,y_2)}\,dx.$$

Ex. $y'' + 2y' + y = e^{-x}\sin(x) = R(x)$

1. hom. sol^{n}: $\dfrac{-2 \pm \sqrt{4-4\cdot1\cdot1}}{2\cdot1} = -1$, real repeated

$y_h(x) = c_1y_1 + c_2y_2 = c_1\underbrace{e^{-x}}_{y_1} + c_2\underbrace{xe^{-x}}_{y_2}$

2. Wronskian: $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}$

$= e^{-x}(e^{-x} - \cancel{xe^{-x}}) + \cancel{e^{-x}xe^{-x}}$

$= e^{-2x} \neq 0$ (good news!)

3. Compute u_1 & u_2!

$u_1 = -\displaystyle\int \frac{y_2R}{W(y_1,y_2)}\,dx = -\int \frac{xe^{-x}e^{-x}\sin(x)}{e^{-2x}}$

$= -\displaystyle\int x\sin(x)$ $\qquad u = x \quad dv = \sin(x)$; $du = 1 \quad v = \cos(x)$

$= x\cos(x) - \displaystyle\int \cos(x)$

$= x\cos(x) - \sin(x)$

$u_2 = +\displaystyle\int \frac{y_1R}{W(y_1,y_2)} = \int \frac{\cancel{e^{-x}}e^{-x}\sin(x)}{\cancel{e^{-2x}}} = \int \sin(x)$

$= -\cos(x)$

4. Put together y_p! $\quad y_p = y_1u_1 + y_2u_2$

$= e^{-x}(\cancel{x\cos(x)} - \sin(x))$

$+ \cancel{xe^{-x}(-\cos(x))}$

$= -e^{-x}\sin(x)$

Thus!
$$\boxed{y = y_h + y_p = c_1e^{-x} + c_2xe^{-x} - e^{-x}\sin(x)}$$

Ex. $y'' + y = \tan(x)$ $\qquad$ Cannot be done with undet. coeff's!

1. hom. sol^{n}! $\quad r^2 + 1 = 0$

$r = \pm i$ $\qquad y_h(x) = c_1\underbrace{\cos(x)}_{y_1} + c_2\underbrace{\sin(x)}_{y_2}$

$y_1 = \cos(x)$

$y_2 = \sin(x)$

2. Wronskian! $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$

$= \cos^2(x) + \sin^2(x)$

$= 1$ (nice!!)

3. Solve for u_1 & u_2!

$\boxed{u_1 = -\displaystyle\int \frac{y_2R}{W(y_1,y_2)}} = -\displaystyle\int \frac{\sin(x)\tan(x)}{1}\,dx$

$= -\displaystyle\int \frac{\sin^2(x)}{\cos(x)}\,dx$ $\qquad$ use $\sin^2(x) = 1 - \cos^2(x)$

$= -\displaystyle\int \frac{(1-\cos^2(x))}{\cos(x)}$

$= -\displaystyle\int \sec(x) - \cos(x) = \int \cos(x) - \sec(x)$

$= \sin(x) - \ln|\sec(x) + \tan(x)|$

$\boxed{u_2 = +\displaystyle\int \frac{y_1R}{W(y_1,y_2)}\,dx} = \displaystyle\int \frac{\cos(x)\tan(x)}{1}\,dx$

$= \displaystyle\int \cancel{\cos(x)}\frac{\sin(x)}{\cancel{\cos(x)}}$

$= \displaystyle\int \sin(x)$

$= -\cos(x).$

4. Put it together! $\quad \boxed{y_p = y_1u_1 + y_2u_2}$

$= \cos(x)(\cancel{\sin(x)} - \ln|\sec(x) + \tan(x)|)$

$+ \sin(x)(\cancel{-\cos(x)})$

$= -\cos(x)\ln|\sec(x) + \tan(x)|$

5. Final sol^{n}!
$$\boxed{\begin{aligned} y &= y_h + y_p \\ &= c_1\cos(x) + c_2\sin(x) \\ &\quad - \cos(x)\ln|\sec(x) + \tan(x)| \end{aligned}}$$