EL21 Lab 3 live

Thursday, February 4, 2021   9:42 AM

- Wronskian
- Undetermined coefficients

Ex. Compute the wronskian of the solution set
$y_1(t) = e^{r_1 t}$, $y_2(t) = e^{r_2 t}$.
Show when they form a fundamental solution set.

Recall: $W(y_1,y_2)(t_0) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= y_1 y_2' - y_1' y_2$

$y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$
$y_1' = r_1 e^{r_1 t}$, $y_2' = r_2 e^{r_2 t}$.

$W(y_1,y_2)(t_0) = r_2 e^{r_1 t_0} e^{r_2 t_0} - r_1 e^{r_1 t_0} e^{r_2 t_0}$

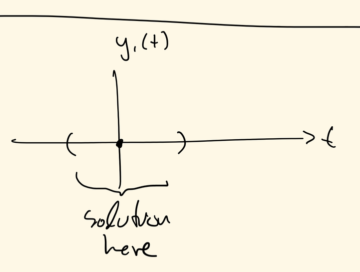
$= (r_2 - r_1) \underbrace{e^{(r_1+r_2)t_0}}_{\text{if this} \neq 0}$ (choose $t_0 = 0$)

$= (r_2 - r_1)$

if $r_2 \neq r_1$, then $W(y_1,y_2)(0) \neq 0$ $\Rightarrow$ they form a fundamental solution set.

if $r_1 = r_2$, then they do not form a fund. sol^{n} set.

Ex. Can $y_1(t) = \sin(t^2)$ be a solution on an interval containing the origin of a second order diff^{l} eq^{n} with continuous coefficients?



Sol^{n}: Suppose $y_1(t) = \sin(t^2)$ and $y_2(t)$ form a fund. sol^{n} set.

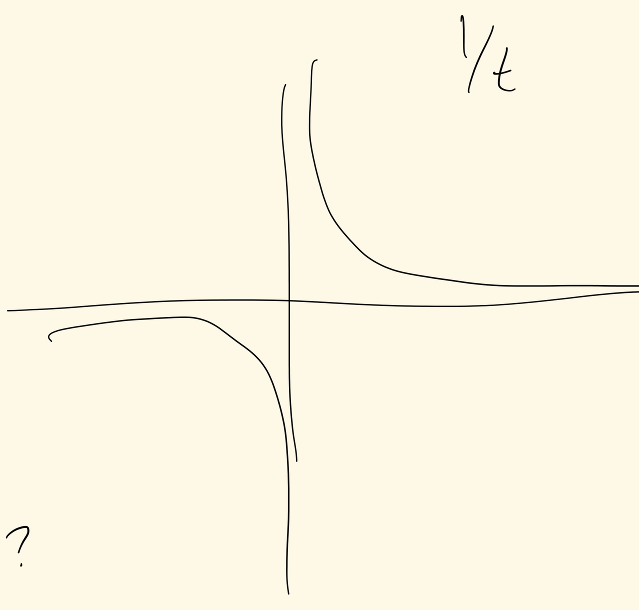
$y_1'(t) = \cos(t^2)\cdot 2t = 2t\cos(t^2)$

$W(y_1,y_2)(0) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ → not diff^{l} or continuous at $t=0$!

$= [\sin(t^2) y_2' - y_2 \cdot 2t\cos(t^2)]\big|_{t=0}$

$= 0\, y_2' - y_2 \cdot 0 = 0$

No! It cannot be a piece of a fund. sol^{n} set.



What if $y_2 = 1/t$?

$- y_2 \cdot 2t\cos(t^2)$

$= -\frac{1}{t} \cdot 2t\cos(t^2)$

$= -2\cos(t^2)\big|_{t=0}$

$= -2 < 0$

Undetermined Coefficients:

solve: $ay'' + by' + cy = \underbrace{Q(x)}$.

based on its form, guess a solution and solve for some coefficients.

Ex. Solve $y'' + y' - 2y = x^2$ → inhomogeneous!

1. Homogeneous sol^{n} (Characteristic eq^{n})
2. Particular sol^{n} (Undetermined coeff's)

1. $r^2 + r - 2 = 0$

$\hookrightarrow \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2(1)} = \frac{-1}{2} \pm \frac{\sqrt{9}}{2}$

$= -\frac{1}{2} \pm \frac{3}{2}$ → $r_1 = 1$, $r_2 = -2$

$\Rightarrow$ two real, distinct roots

So, $\boxed{y_h(x) = c_1 e^{x} + c_2 e^{-2x}}$

2. Find the particular sol^{n}! Look at $Q(x)$: $Q(x) = x^2$.
$\Rightarrow$ polynomial
$\Rightarrow$ second order.

So, guess: $y_p(x) = Ax^2 + Bx + C$, and solve for A, B & C.

So, plug in $y_p(x)$ into the eq^{n}:

$y_p = Ax^2 + Bx + C$
$y_p' = 2Ax + B$
$y_p'' = 2A$

$Q = x$
$\hookrightarrow$ guess $Ax + B$

$Q = x^4$,
guess: $Ax^4 + Bx^3 + Cx^2 + Dx + E$

Plug in:
$x^2 = 2A + (2Ax + B) - 2(Ax^2 + Bx + C)$
$= (-2A)x^2 + (2A - 2B)x + (2A + B - 2C)$

Match the coefficients on each side!

$1\cdot x^2 + 0\cdot x + 0 = (-2A)x^2 + (2A-2B)x + (2A + B - 2C)$

solve: $\begin{cases} -2A = 1 \\ 2A - 2B = 0 \\ 2A + B - 2C = 0 \end{cases}$

$-2A = 1 \rightarrow \boxed{A = -1/2}$

$2A - 2B = 0 \rightarrow \boxed{B = A = -1/2}$

$C = A + \frac{B}{2}$
$= -\frac{1}{2} + \left(-\frac{1}{4}\right)$
$= -\frac{2}{4} - \frac{1}{4}$
$= -3/4$

Thus: $y_p(x) = Ax^2 + Bx + C$
$= -\frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}$

Our final sol^{n} is then given by:

$$\boxed{\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1 e^{x} + c_2 e^{-2x} + \left(-\frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}\right) \end{aligned}}$$

Ex: Solve $2y'' + 4y' + 2y = e^{-x}$.

1. hom. sol^{n}: $\frac{-4 \pm \sqrt{16 - 4(2)(2)}}{2(2)} = -1$.
→ one repeated root.

$y_h(x) = c_1 e^{-x} + c_2 x e^{-x}$.

2. $Q(x) = e^{-x}$. Guess $y_p(x) = \cancel{Ae^{-x}}$.
this is the homogeneous sol^{n}!!

$e^{-x} = 2Ae^{-x}x$
$\boxed{A = 0}$

Okay, multiply by x! Guess: $y_p(x) = Axe^{-x}$
same problem!

Guess: $\cancel{(Ax + B)e^{-x}}$
$= Axe^{-x} + Be^{-x}$

So! Try: $y_p(x) = (Ax^2 + Bx + C)e^{-x}$.
$= Ax^2e^{-x} + \cancel{Bxe^{-x}} + \cancel{Ce^{-x}}$ → throw away
$=0$ $=0$

Plug into the eq^{n}:

$1 e^{-x} = 2(2Ae^{-x} - 2Axe^{-x} - 2Axe^{-x} + Ax^2e^{-x})$
$+ 4\underbrace{(2Axe^{-x} - Ax^2e^{-x})}_{y_p'}$
$+ 2\underbrace{(Ax^2e^{-x})}_{y_p}$

$= 2(Ax^2e^{-x} - 4Axe^{-x} + 2Ae^{-x})$
$+ 4(-Ax^2e^{-x} + 2Axe^{-x})$
$+ 2(Ax^2e^{-x})$

$= x^2\underbrace{(2A - 4A + 2A)}_{=0}e^{-x}$
$x\underbrace{(-8A + 8A)}_{=0}e^{-x}$
$1(4A)e^{-x}$

$\Rightarrow 1 = 4A$
$\Rightarrow A = 1/4$.

So, $y_p(x) = \frac{1}{4}x^2e^{-x}$.

So, our final sol^{n} is:

$$\boxed{\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{4}x^2 e^{-x} \end{aligned}}$$