EL21 Lab 2 live

Thursday, January 28, 2021 9:58 AM

Ex. $t^2 \frac{dy}{dt} + 2ty = y^3$ $\boxed{y^n}$ (Bernoulli equation)

1. Standard form! $\underbrace{1 \cdot \frac{dy}{dt} + \underbrace{\frac{2}{t}}_{P(t)} y = \underbrace{\frac{1}{t^2}}_{Q(t)} y^3}$ $\to n = 3$

2. Use the substitution: $\boxed{u = y^{1-n}}$ ⟶ turns an equation for y into $\boxed{\frac{du}{dt} + (1-n)P(t)u = (1-n)Q(t)}$ an equation for u!

$y = u^{\frac{1}{1-n}}$

So, solve

$$\frac{du}{dt} + (1-3)\cdot\frac{2}{t} u = (1-3)\cdot\frac{1}{t^2}$$

$$\frac{du}{dt} \underbrace{- \frac{4}{t}}_{p(t)} u = \underbrace{-\frac{2}{t^2}}_{q(t)}$$

3. Solve using int. fact. $\Rightarrow \mu(t) = e^{\int p(t)} = e^{-\int \frac{4}{t}} = e^{-4\ln(t)} = e^{\ln(t^{-4})} = \boxed{t^{-4}}$

Multiply the whole eq^n by t^{-4}:

$$t^{-4}\frac{du}{dt} - 4t^{-5}u = -2t^{-6}$$

$$\frac{d}{dt}\left(t^{-4}u\right) = -2t^{-6}$$

integrate both sides:

$$t^{-4}u = -2\int t^{-6} = -2\left(-\frac{1}{5}t^{-5}\right) + C$$

$$= \frac{2}{5}t^{-5} + C$$

So! $\boxed{u(t) = \frac{2}{5}t^{-1} + Ct^4}$

Then, write the solution $y(t)$!

$$y(t) = u^{\frac{1}{1-n}} = u^{-1/2}$$

$$= \left(\frac{2}{5}t^{-1} + Ct^4\right)^{-1/2}$$

$$\boxed{y(t) = \frac{1}{\sqrt{\frac{2}{5}t^{-1} + Ct^4}}}$$

Ex. $(3xy + y^2) + (x^2 + xy)y' = 0$, $y' \to \frac{dy}{dx}$

$$(3xy + y^2)dx + (x^2 + xy)dy = 0$$

$$\frac{\partial M}{\partial y} = 3x + 2y \neq \frac{\partial N}{\partial x} = 2x + y$$

Not exact!

But! We can make it exact by multiplying the equation by $\mu(x)$ and solving for μ.

$$\underbrace{\mu(x)(3xy + y^2)}_{M(x,y)}dx + \underbrace{\mu(x)(x^2 + xy)}_{N(x,y)}dy = 0$$

$$\frac{\partial M}{\partial y} = \mu(x)[3x + 2y] \quad , \quad \frac{\partial N}{\partial x} = \mu'(x)(x^2+xy) + \mu(x)(2x+y)$$

Using that we require $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$,

$$\longrightarrow \mu(x)[3x + 2y] = \mu'(x)(x^2+xy) + \mu(x)(2x+y)$$

rearrange:

$$\mu'(x)\underbrace{(x^2+xy)}_{N(x,y)} = \mu(x)[\underbrace{3x+2y}_{\frac{\partial M}{\partial y}} - \underbrace{(2x+y)}_{\frac{\partial N}{\partial x}}]$$

$$\frac{\mu'(x)}{\mu(x)} = \frac{x+y}{x^2+xy} = \frac{\cancel{x+y}}{x\cancel{(x+y)}} = \frac{1}{x}$$

$$\boxed{\frac{\mu'}{\mu} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N(x,y)}}$$

So, now solve:

$$\boxed{\frac{\mu'(x)}{\mu(x)}} = \frac{1}{x}$$ → did not depend on y!

$$\int \frac{d}{dx}\left(\ln(\mu(x))\right) = \int \frac{1}{x}$$

$$\ln(\mu(x)) = \ln(x)$$

$$\boxed{\mu(x) = x}$$

Back to the Original eq^n:

$$x(3xy + y^2)dx + x(x^2 + xy)dy = 0$$

$$(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0$$

$$\frac{\partial M}{\partial y} = 3x^2 + 2xy = \frac{\partial N}{\partial x} = 3x^2 + 2xy$$

Now solve as an exact equation!

Second order, linear, const. coeff. eq^n's

Consider the following equation:

$$ay'' + by' + cy = 0$$ → homogeneous.

Solving this depends on the roots of the characteristic equation.

Guess that $y(x) = e^{rx}$.

$$ay'' + by' + cy = e^{rx}\underbrace{(ar^2 + br + c)}_{=0} = 0$$

Char eq^n: $ar^2 + br + c = 0$

Roots: $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Call these two roots r_1 and r_2.

Solutions are then of the form $\begin{cases} y_1 = c_1 e^{r_1 x} \\ y_2 = c_2 e^{r_2 x} \end{cases}$

There are 3 cases:

(I) $b^2 - 4ac > 0 \Rightarrow$ two real, distinct roots. $(r_1, r_2 \in \mathbb{R})$

(II) $b^2 - 4ac = 0 \Rightarrow$ one real, repeated root. $(r_1 = r_2 \in \mathbb{R})$

(III) $b^2 - 4ac < 0 \Rightarrow$ two complex roots, which are conjugate! $\left[\begin{array}{l} r_1 = \alpha + i\beta, \\ r_2 = \alpha - i\beta. \end{array}\right.$

Ex. Solve $2y'' + 4y' - y = 0$

Char eq^n: $2r^2 + 4r - 1 = 0$

roots: $\frac{-4 \pm \sqrt{16 - 4(2)(-1)}}{2 \cdot (2)} = -1 \pm \frac{\sqrt{24}}{4}$

$$= -1 \pm \frac{\sqrt{6}}{2}$$

Since $b^2 - 4ac > 0$, two real roots

$$r_1 = -1 + \frac{\sqrt{6}}{2},$$

$$r_2 = -1 - \frac{\sqrt{6}}{2}.$$

Final sol^n: $y(x) = c_1 e^{\left(-1+\frac{\sqrt{6}}{2}\right)x} + c_2 e^{\left(-1-\frac{\sqrt{6}}{2}\right)x}$.

Ex. Solve $2y'' + 4y' + 2y = 0$

Char Eq^n: $2r^2 + 4r + 2 = 0$

Roots: $\frac{-4 \pm \sqrt{16 - 4(2)(2)}}{2(2)} = -1$, $16 - 4(2)(2) = 0$

One real, repeated root!

$$y(x) = \cancel{c_1 e^{-1x} + c_2 e^{-1x}}$$

$$= \cancel{(c_1 + c_2)e^{-x} = Ce^{-x}}.$$

So, when we have a repeated root, we find another solution (in addition to c_1e^{-x}) by multiplying by x. So,

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

Ex. Solve $\frac{1}{2}y'' + y' + 5y = 0$

Char eq^n: $\frac{1}{2}r^2 + r + 5 = 0$

Roots: $\frac{-1 \pm \sqrt{1 - 4(\frac{1}{2})(5)}}{\cancel{2(\frac{1}{2})} = 1} = -1 \pm \sqrt{-9}$

$$= -1 \pm i3 \rightsquigarrow \alpha \pm i\beta$$

So, complex conjugates with $\alpha = -1$, $\beta = 3$.

So, our solution is:

$$y(x) = \boxed{e^{\alpha x}\left[c_1\cos(\beta x) + c_2\sin(\beta x)\right]}$$

$$= e^{-x}\left[c_1\cos(3x) + c_2\sin(3x)\right]$$

$$ay'' + by' + cy = Q(x)$$

$= 0$ first

$= Q(x)$ second