EL18 Lab 9 - live

Tuesday, November 3, 2020 5:56 PM

Topics: Triple integrals!

Ex. $$\int_0^1 \int_x^{2x} \int_0^y 2xyz \, dz\, dy\, dx$$

$$\int_0^y 2xyz\, dz = 2xy \int_0^y z\, dz = 2xy \cdot \left(\frac{z^2}{2} \Big|_0^y \right) = \cancel{2}xy \cdot \frac{y^2}{\cancel{2}}$$

$$\int_0^1 \int_x^{2x} xy^3 \, dy\, dx = \int_0^1 x \left(\frac{y^4}{4} \Big|_{y=x}^{y=2x} \right) dx = \int_0^1 \frac{x}{4} \left((2x)^4 - x^4 \right) dx$$

$$= \int_0^1 \frac{15x^5}{4} dx = \frac{15}{4} \int_0^1 x^5 dx = \frac{15}{4} \cdot \frac{1}{6} x^6 \Big|_0^1 = \frac{15}{24} = \boxed{\frac{5}{8}}$$

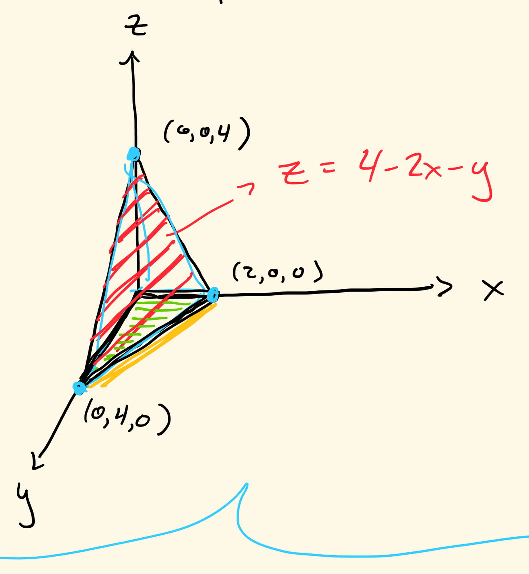
Ex. Find the volume of the tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

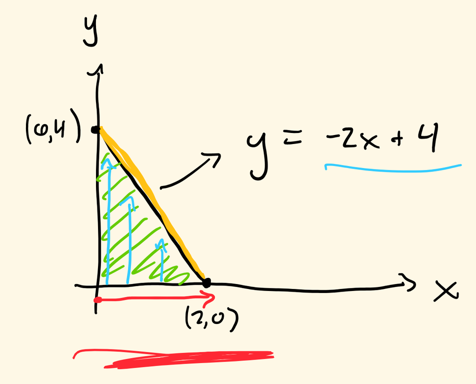
So, $z = 4 - 2x - y$

when $(x,y) = 0$, $z = 4$

$(x,z) = 0$, $y = 4$

$(y,z) = 0$, $x = 2$





So, $R = \{(x,y,z) : 0 \le x \le 2,\ 0 \le y \le -2x+4,\ 0 \le z \le 4-2x-y\}$.

So, $$\text{Volume} = \iiint_R dV = \int_0^2 \int_0^{-2x+4} \int_0^{4-2x-y} 1 \cdot dz\, dy\, dx$$

$$= \int_0^2 \int_0^{-2x+4} \left(z \Big|_{z=0}^{z=4-2x-y} \right) dy\, dx$$

$$= \int_0^2 \int_0^{-2x+4} (4-2x-y)\, dy\, dx = \int_0^2 \left((4-2x)y - \frac{y^2}{2} \right) \Big|_{y=0}^{y=-2x+4} dx$$

$$= \int_0^2 \left((4-2x)(-2x+4) - \frac{(-2x+4)^2}{2} \right) dx$$

$$= \frac{1}{2} \int_0^2 (4-2x)^2 dx \quad , \text{ set } u = 4-2x, \ -\frac{1}{2} du = \cancel{-2}dx$$

$$= \frac{1}{2}\left(-\frac{1}{2}\right) \int u^2 du$$

$$= -\frac{1}{4} \frac{u^3}{3} \Big|_{x=0}^{x=2} \longrightarrow -\frac{1}{4} \cdot \frac{1}{3} \cdot (4-2x)^3 \Big|_0^2$$

$$= \frac{1}{4} \cdot \frac{1}{3} \cdot 4^3 = \frac{4^2}{3} = \boxed{\frac{16}{3}}$$

Ex. Use cyl. coords. to find the volume of the region lying in the first octant, bounded above by

$z = 4 - x^2 - y^2$

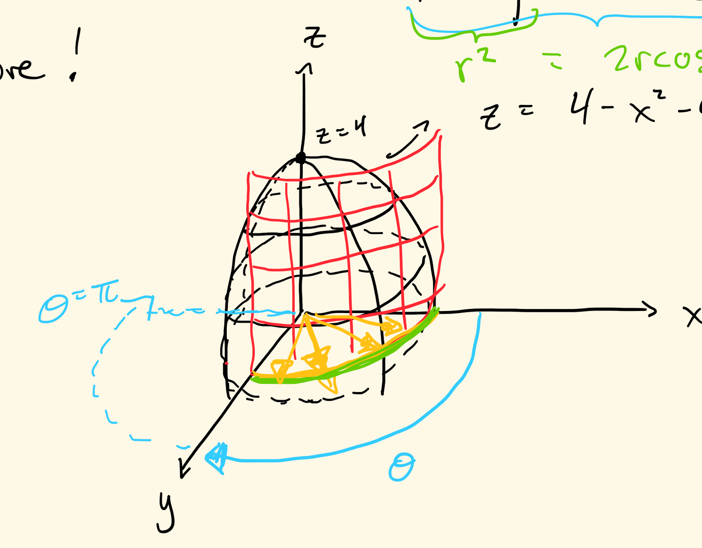
and

$x^2 + y^2 = 2x$ $\rightsquigarrow$ $r^2 = 2x = 2r\cos\theta$, $r = 2\cos\theta$

$y^2 = 2x - x^2$, $y = \pm\sqrt{2x - x^2}$

$r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta$

First, picture!



$r \le ?$

Recall: $z = z$, $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$.

$r^2 = 2x$

$r^2 = 2r\cos\theta$

$r \le 2\cos\theta$

So, $0 \le z \le 4 - x^2 - y^2 = 4 - r^2$

$0 \le r \le 2\cos\theta$

$0 \le \theta \le \pi/2$ $\Big\}$ E

$$\text{Volume} = \iiint_E dV = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_0^{4-r^2} dz\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r(4-r^2)\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} (4r - r^3)\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \left(2r^2 - \frac{r^4}{4} \right) \Big|_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(2(2\cos\theta)^2 - \frac{(2\cos\theta)^4}{4} \right) d\theta$$

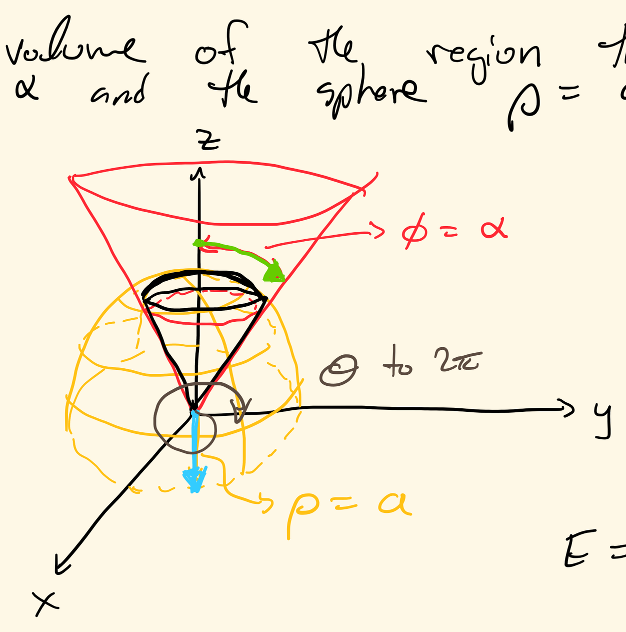
$$= \int_0^{\frac{\pi}{2}} (8\cos^2\theta - 4\cos^4\theta)\, d\theta \quad \text{using } \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\vdots$$

$$\boxed{\frac{5\pi}{4}}$$

Ex. Find the volume of the region that lies inside the cone $\phi = \alpha$ and the sphere $\rho = a$.

Picture:



$E = \{(\rho, \phi, \theta) : 0 \le \rho \le a,\ 0 \le \phi \le \alpha,\ 0 \le \theta \le 2\pi\}$.

$$\text{Volume} = \iiint_E dV = \iiint_E dx\,dy\,dz$$

$$= \int_0^{2\pi} \int_0^{\alpha} \int_0^{a} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\alpha} \sin\phi \left(\frac{\rho^3}{3} \Big|_0^a \right) d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\alpha} \sin\phi\, \frac{a^3}{3}\, d\phi\, d\theta$$

$$= \frac{a^3}{3} \int_0^{2\pi} \left(-\cos\phi \Big|_{\phi=0}^{\phi=\alpha} \right) d\theta$$

$$= \frac{a^3}{3} \cdot (1 - \cos(\alpha)) \int_0^{2\pi} d\theta$$

$$= \boxed{\frac{2\pi}{3} a^3 (1 - \cos\alpha)} \approx 0$$

≈ 1