EL18 Lab 8 - live

Tuesday, October 27, 2020 5:51 PM

Ex. A thin plate is lying in the xy-plane and is bounded by the curves:

$$x^2 + 4y^2 = 12, \quad \rightsquigarrow \quad y = \pm \tfrac{1}{2}\sqrt{12-x^2}$$

and

$$x = 4y^2, \quad \rightsquigarrow \quad y = \pm \tfrac{1}{2}\sqrt{x}$$

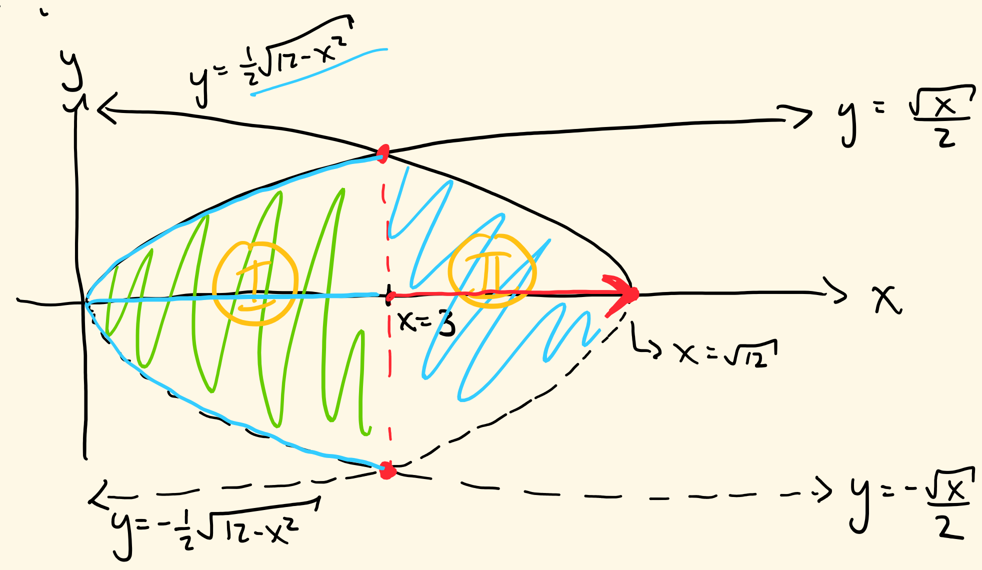
with mass density $\rho(x,y) = x$.

What is the total mass of the plate?

Recall:

$$m = \iint_D \rho(x,y)\,dx\,dy$$

First, draw a picture!



When does: $\cancel{\tfrac{1}{2}}\sqrt{12-x^2}^{\,\cancel{}} = \cancel{\tfrac{1}{2}}\sqrt{x}$

$\hookrightarrow \; 12 - x^2 = x$

$\hookrightarrow \; x^2 - x + 12 = 0 \implies (x+4)(x-3) = 0$

We want $x = 3 > 0$

Region (I): $D_1 = \left\{(x,y) : -\frac{\sqrt{x}}{2} \le y \le \frac{\sqrt{x}}{2},\ 0 \le x \le 3\right\}$

Region (II): $D_2 = \left\{(x,y) : -\frac{1}{2}\sqrt{12-x^2} \le y \le \frac{1}{2}\sqrt{12-x^2},\ 3 \le x \le \sqrt{12}\right\}$

So, $m = \iint_D \rho(x,y)\,dy\,dx = \iint_{D_1} x\,dy\,dx + \iint_{D_2} x\,dy\,dx$

$$= \int_0^3 \int_{-\frac{\sqrt{x}}{2}}^{\frac{\sqrt{x}}{2}} x\,dy\,dx + \int_3^{\sqrt{12}} \int_{-\frac{1}{2}\sqrt{12-x^2}}^{\frac{1}{2}\sqrt{12-x^2}} x\,dy\,dx$$

$$= \int_0^3 x \left(y \Big|_{y=-\frac{\sqrt{x}}{2}}^{y=\frac{\sqrt{x}}{2}} \right) dx + \int_3^{\sqrt{12}} x \left(y \Big|_{y=-\frac{1}{2}\sqrt{12-x^2}}^{y=\frac{1}{2}\sqrt{12-x^2}} \right) dx$$

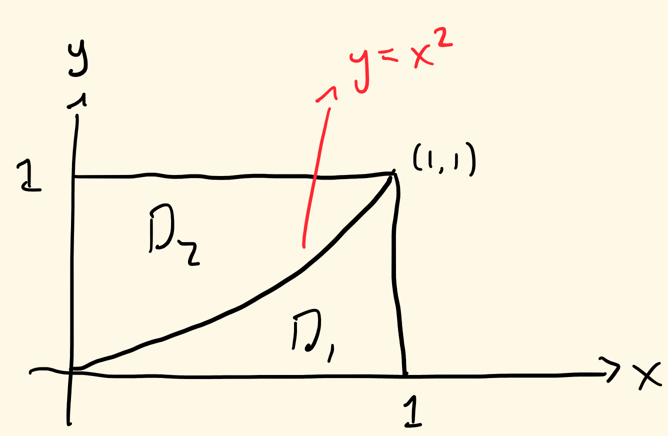
$$= \int_0^3 \underbrace{x \cdot \sqrt{x}}_{x^{3/2}}\,dx + \int_3^{\sqrt{12}} x \cdot \sqrt{12-x^2}\,dx, \quad \text{set} \quad u = 12 - x^2, \quad -\frac{du}{2} = \cancel{2}x\,dx$$

$$= \frac{2}{5} x^{5/2} \Big|_{x=0}^{x=3} + \int_3^{\sqrt{12}} \left(-\tfrac{1}{2}\right)\sqrt{u}\,du$$

$$= \frac{2}{5}(3)^{5/2} - \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_{x=3}^{x=\sqrt{12}}$$

$$= \frac{2}{5} 3^{5/2} - \frac{1}{3}\left((12-x^2)^{3/2}\right)\Big|_{x=3}^{x=\sqrt{12}}$$

$$= \frac{2}{5} 3^{5/2} + \frac{1}{3}(3)^{3/2} = \boxed{\frac{23\sqrt{3}}{5}}$$



$\Longleftrightarrow$