EL18 Lab 11 - live

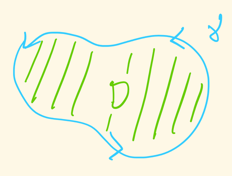
Tuesday, November 24, 2020 4:34 PM

Topics: - Green's theorem (+ more "difficult" parametrization)
- Surface integrals
- Flux

Green's Theorem: Suppose γ is a "positively" oriented closed curve. Then,

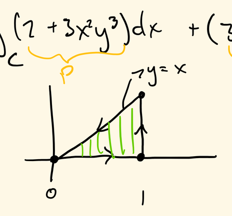
$$\int_\gamma P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA,$$

where D is the region enclosed by γ.



Ex. Compute $\int_C \underbrace{(2+3x^2y^3)}_{P}dx + \underbrace{(3+2x^3y^2)}_{Q}dy$

over



$\left\{\begin{matrix} 0 \le x \le 1 \\ 0 \le y \le x \end{matrix}\right\}$

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$\Rightarrow$ $Q = 3 + 2x^3y^2$, $\frac{\partial Q}{\partial x} = 6x^2y^2$

$P = 2 + 3x^2y^3$, $\frac{\partial P}{\partial y} = 9x^2y^2$

So,

$$\int_C P\,dx + Q\,dy = \int_0^1\int_0^x (6x^2y^2 - 9x^2y^2)\,dy\,dx$$

$$= -3\int_0^1\int_0^x x^2y^2\,dy\,dx$$

$$= -\cancel{3}\int_0^1 x^2\left(y^3/\cancel{3}\Big|_{y=0}^{y=x}\right)dx$$

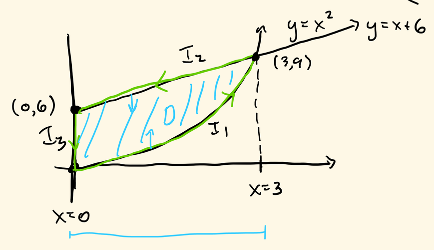
$$= -\int_0^1 x^2x^3\,dx = -\frac{1}{6}x^6\Big|_0^1$$

$$= \boxed{-\frac{1}{6}}$$

Since we want clockwise,

$$\int_C P\,dx + Q\,dy = -\left(-\frac{1}{6}\right) = \boxed{\frac{1}{6}}$$

Ex. Compute $\int_C (2+3x^2y^3)dx + (3+2x^3y^2)dy$ over the curve given by:
$\left\{\begin{matrix} y = x^2 \\ y = x+6 \\ x = 0 \end{matrix}\right.$, counter clockwise.



Parametrize: I_1: $x = t$, $y = t^2$, $0 \le t \le 3$

$\boxed{r(t) = \langle t, t^2\rangle}$, $t=0$, $r(0) = (0,0)$

$t = 3$, $r(3) = \langle 3, 9\rangle$

When does $x^2 = x+6$?

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$\Rightarrow \boxed{x = 3}$

I_2: Starting point : $(3,9)$
End point : $(0,6)$

$\underbrace{t(0,6)}_{\text{finish}} + \underbrace{(1-t)(3,9)}_{\text{start}}$ for $\boxed{0 \le t \le 1}$

Take $t=0$: $r(0) = (3,9)$
$t=1$: $r(1) = (0,6)$

$(0, 6t) + (3(1-t), 9(1-t))$

$= \boxed{(3(1-t), 9-3t)} =: r(t)$

I_3: $x = 0$, y starts at $1 \to 0$.

$\boxed{r(t) = \langle 0, 1-t\rangle}$, $0 \le t \le 1$.

I_1: $r(t) = \langle x(t), y(t)\rangle = \langle t, t^2\rangle$, $0 \le t \le 3$.

$\langle \frac{dx}{dt}, \frac{dy}{dt}\rangle = \langle 1, 2t\rangle$

$$\int_0^3 (2 + 3\cdot(t)^2\cdot(t^2)^3)\cdot 1\cdot dt + (3 + 2(t)^3(t^2)^2)\cdot 2t\,dt$$

$$= \int_0^3 (2+3t^8)dt + (6t + 4t^8)dt$$

$$= \left(2t + \frac{3t^9}{9} + \frac{6t^2}{2} + \frac{4t^9}{9}\right)\Big|_0^3 = 2\cdot 3 + \frac{3^9}{3} + \frac{6\cdot 9}{2} + \frac{4\cdot 3^9}{9}$$

$$= \boxed{6 + 3^8 + 3^3 + 4\cdot 3^7} = A$$

I_2: $r(t) = \langle x(t), y(t)\rangle = \langle 3(1-t), 9-3t\rangle$

$\langle \frac{dx}{dt}, \frac{dy}{dt}\rangle = \langle -3dt, -3dt\rangle$

$$\int_0^1 (2 + 3(3(1-t))^2(9-3t)^3)(-3)dt + (3 + 2(3(1-t))^3(9-3t)^2)(-3)dt$$

$$= -3\int_0^1 (2 + 3^3(1-t)^2(9-3t)^3)dt + (3 + 2(3(1-t))^3(9-3t)^2)dt = \boxed{?} = B$$

I_3: $r(t) = \langle 0, 1-t\rangle$, $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = -1$

$$\int_0^1 (2+0)\cdot 0 + (3+0)(-1)dt = -3t\Big|_0^1 = \boxed{-3} =$$

$$\int_C = A + B - 3$$

Suppose we have a surface $z = g(x,y)$

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dA,$$

where D is the domain for $g(x,y)$ in the (x,y)-plane.

Ex. Find the surface area of the portion of the paraboloid $x^2 + y^2 = 6z$ (I) inside the sphere $x^2+y^2+z^2 = 16 = 4^2$ (II)

$\iint_S dS$

$r^2 = x^2+y^2$, $0 \le x^2+y^2 \le 16 - z^2$
$r \le 16 - z^2$

$f(x,y,z) = 1$.

Now, solve for z: $\boxed{z = \frac{x^2+y^2}{6} =: g(x,y)},$

(II) - (I): $z^2 = 16 - 6z$

$z^2 + 6z - 16 = 0$

$(z+8)(z-2) = 0$

$\boxed{z = 2}$

D is given by $x^2 + y^2 \le 4^2$

Next,

$$\boxed{\iint_S dS} = \iint_D \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dA, \quad \frac{\partial g}{\partial x} = \frac{x}{3}, \quad \frac{\partial g}{\partial y} = \frac{y}{3}$$

$$= \iint_D \sqrt{1 + \frac{x^2+y^2}{9}}\,dA$$

$D = \{(r,\theta) : 0 \le \theta \le 2\pi, 0 \le r \le 16 - z^2 = \boxed{12}\}$

$$= \int_0^{2\pi}\int_0^{12} \sqrt{1 + \frac{r^2}{9}}\,r\,dr\,d\theta,$$

set $u = 1 + \frac{r^2}{9}$, $\frac{9}{2}du = \frac{\cancel{2}r\,dr}{\cancel{9}}$

$$= \int_0^{2\pi}\int_0^{12} \sqrt{u}\cdot\frac{9}{2}\,du\,d\theta$$

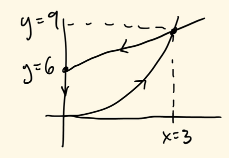
$$= \int_0^{2\pi} \frac{\cancel{9}\,3}{\cancel{2}}\left(u^{3/2}\cdot\frac{\cancel{2}}{\cancel{3}}\Big|_{r=0}^{r=12}\right)d\theta = \int_0^{2\pi} 3\left(1 + \frac{r^2}{9}\right)^{3/2}\Big|_{r=0}^{r=12} d\theta$$

$$= 3\left[\left(1 + \frac{144}{9}\right)^{3/2} - 1\right]\cdot\int_0^{2\pi} d\theta$$

$$= 6\pi\left[\left(1 + \frac{144}{9}\right)^{3/2} - 1\right]$$

$$= \boxed{6\pi\left(\left(\frac{21}{9}\right)^{3/2} - 1\right)}.$$

Prepare: - Surface integrals (form $g(x,y) = z$)
- Flux through a surface (through a cylinder)



$\left.\boxed{\begin{matrix} y = x^2 \\ y = x+6 \\ x = 0 \end{matrix}}\right)$

$x^2 = ax + 1$
$x^2 - ax - b = 0$
$(x+2)(x-3) = 0$
$x^2 - x - 6 = 0$
$x^2 = ax + b$

I_1: $x = t$, $y = t^2 \Rightarrow r(t) = \langle t, t^2\rangle$, $0 \le t \le 3$

I_2: $x = t$, $y = t+6 \Rightarrow r(t) = \langle t, t+6\rangle$, wrong way, so take $t \mapsto t+3$,

$r(t) = \langle t+3, t+9\rangle$
$r(0) = \langle 3, 9\rangle$
$r($

$(3,9) \longrightarrow (0,6)$, take

$(1-t)(3,9) + t(0,6)$

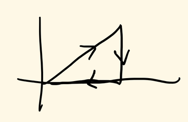
$= (3(1-t), 9(1-t) + 6t) = (3(1-t), 9-3t)$

$t=0$: $(3,9)$, $t=1$, $(0,6)$

I_3: $x = 0$, $y = 6-t$, $0 \le t \le 6$.

$$\int_C (2+3x^2y^3)dx + (3+2x^3y^2)dy$$

over



Using:

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int u\,dv = uv - \int v\,du$$