EL06 Lab 9 - live

Monday, November 16, 2020 3:59 PM

Topics - line integrals:

Suppose you're given a curve C in $\mathbb{R}^2$ given by

$$r(t) = \langle x(t), y(t) \rangle, \quad \alpha \le t \le \beta$$

The line integral w.r.t. arc length of a function $f(x,y)$ over C is given by:

$$\int_C f(x,y)\,ds = \int_\alpha^\beta f(x(t),y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

If we are given $F(x,y) = \langle P(x,y), Q(x,y) \rangle$, then

$$\int_C F\cdot dr = \int_C F\cdot T(s)\,ds = \boxed{\int_C (P\,dx + Q\,dy)},$$

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

Ex. Evaluate $\int_C F\cdot dr$, where $F(x,y) = \boxed{\frac{\langle x, y\rangle}{(x^2+y^2)^{3/2}}}$ along

C: $\boxed{r(t) = e^t \langle \cos(t), \sin(t) \rangle}$ for $0 \le t \le 1$.

$$\Rightarrow \boxed{\begin{matrix} x(t) = e^t\cos(t) \\ y(t) = e^t\sin(t)\end{matrix}}$$

First:

$$r'(t) = \langle e^t\cos(t) - e^t\sin(t),\ e^t\sin(t) + e^t\cos(t) \rangle$$

$$= e^t\langle \cos(t) - \sin(t),\ \sin(t) + \cos(t)\rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle$$

$$\int_C f(x,y)\,dx = \int_\alpha^\beta f(x(t),y(t))\frac{dx}{dt}\cdot dt \quad \to P$$

$$\int_C f(x,y)\,dy = \int_\alpha^\beta f(x(t),y(t))\frac{dy}{dt}\,dt \quad \to Q$$

Then, $\int_C F\cdot dr = \int_C (P\,dx + Q\,dy) \longrightarrow P(x,y) = \frac{x}{(x^2+y^2)^{3/2}}$

$$Q(x,y) = \frac{y}{(x^2+y^2)^{3/2}}$$

$x(t) = e^t\cos(t)$, $y(t) = e^t\sin(t)$, so

$x^2+y^2 = e^{2t}(\cos^2(t) + \sin^2(t)) = e^{2t}$, hence

$\boxed{(x^2+y^2)^{3/2}} = e^{3t}$.

So,

$$\int_C F\cdot dr = \int_0^1 \left(\frac{x}{(x^2+y^2)^{3/2}}\,dx + \frac{y}{(x^2+y^2)^{3/2}}\,dy \right) dt$$

$$= \int_0^1 \left(\frac{e^t\cos(t)}{e^{3t}}\cdot\frac{dx}{dt} + \frac{e^t\sin(t)}{e^{3t}}\,\frac{dy}{dt} \right) dt$$

$$= \int_0^1 \left(\frac{\cos(t)}{e^{2t}}\left(e^t(\cos(t) - \sin(t))\right) + \frac{\sin(t)}{e^{2t}}\left(e^t(\sin(t) + \cos(t))\right)\right) dt$$

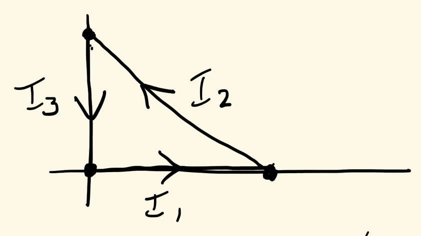
$$= \int_0^1 e^{-t}\left[\underbrace{\cos^2(t) - \cancel{\cos(t)\sin(t)} + \sin^2(t)}_{=1} + \cancel{\sin(t)\cos(t)}\right] dt$$

$$= \int_0^1 e^{-t}\,dt = -e^{-t}\Big|_0^1 = \boxed{\frac{e-1}{e}}$$

Ex. Evaluate $\int_C y^3dx - x^3dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ oriented counter-clockwise!

$F = \langle P, Q\rangle = \langle y^3, -x^3\rangle$

Picture:



Use that $\int_C = \int_{I_1} + \int_{I_2} + \int_{I_3}$.

We must parametrize each I_j, $j = 1,2,3$.

I_1: $y = 0$, $0 \le x \le 1$, so $I_1 = \langle t, 0\rangle$, for $0 \le t \le 1$.

I_2: $y = 1-x$, $0 \le x \le 1$, ~~so $I_2 = \langle t, 1-t\rangle$, for $0 \le t \le 1$~~

wrong! when $t = 0$, $\langle 0, 1\rangle$, when $t = 1$, $\langle 1, 0\rangle$

We can fix this by switching the order of x & y:

$I_2 = \langle 1-t, t\rangle$, $0 \le t \le 1$

$I_2(0) = \langle 1, 0\rangle$
$I_2(1) = \langle 0, 1\rangle$

I_3: $x = 0$, $0 \le y \le 1$, but moving down, so

$I_3 = \langle 0, 1-t\rangle$, $0 \le t \le 1$.

So,

$$\int_C y^3dx - x^3dy = \int_{I_1+I_2+I_3} y^3dx - x^3dy$$

I_1

$$= \int_{I_1} y^3dx - x^3dy$$

In I_1, $\langle x, y\rangle = \langle t, 0\rangle$, $\frac{dy}{dt} = 0$

$$= \int_0^1 0\cdot dx - t^3dy$$

$$= 0$$

I_2

$\int_{I_2} y^3dx - x^3dy$. In I_2,

$\langle x, y\rangle = \langle 1-t, t\rangle$, so

$\left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle = \langle -1dt, 1dt\rangle$. Then,

$$\int_0^1 t^3\cdot(-1dt) - (1-t)^3(1dt)$$

$$= \boxed{-\int_0^1 (t^3 - (1-t)^3)\,dt}$$

$$= -\frac{1}{4} - \frac{1}{4}$$

$$= \boxed{-1/2}$$

I_3

$\int_{I_3} y^3dx - x^3dy$, $\frac{dx}{dt} = 0$

In I_3, $\langle x, y\rangle = \langle 0, 1-t\rangle$

$$= \int_0^1 (1-t)^3\cdot 0 - 0$$

$$= 0$$

So,

$$\int_C y^3dx - x^3dy = \boxed{-\frac{1}{2}}$$

Conservative Vector Field: Given a vector field $F(x)$, if there is a function $f(x)$ so that

$$\nabla f = F,$$

then F is conservative.

Remember: $F = \langle F_1, F_2, \dots, F_n\rangle$. Then,

$f = f(x_1, x_2, \dots, x_n)$

Then, $\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right\rangle$

$$\Rightarrow \boxed{F_i = \frac{\partial f}{\partial x_i}}$$

Ex. Last question had: $F = \langle P, Q\rangle = \langle y^3, -x^3\rangle$

Is F conservative?? $\longrightarrow$ No!!

Does there exist a function $f(x,y)$ so that

$$\frac{\partial f}{\partial x} = y^3 = P$$

$$\frac{\partial f}{\partial y} = -x^3 = Q$$

$$\int^x \frac{\partial f}{\partial x} = f(x,y) = \int^x y^3 = xy^3 + g(y)$$

$$\int^y \frac{\partial f}{\partial y} = f(x,y) = \int^y -x^3 = -yx^3 + h(x)$$

So, in that case: $xy^3 + g(y) = f(x,y) = -yx^3 + h(x)$

So,

$$xy^3 + yx^3 = h(x) - g(y)$$

@ $x = 0$, $= 0$

If $x = 0$, $0 = h(0) - g(y) \Rightarrow g$ is constant.

$y = 0$, $0 = h(y) - g(0) \Rightarrow h(y)$ is constant!

This means that $f(x,y) = xy^3 + C$

$= yx^3 + C$

$$\Rightarrow xy^3 = -yx^3 \longrightarrow\!\!\leftarrow$$