EL06 Lab 8 - live

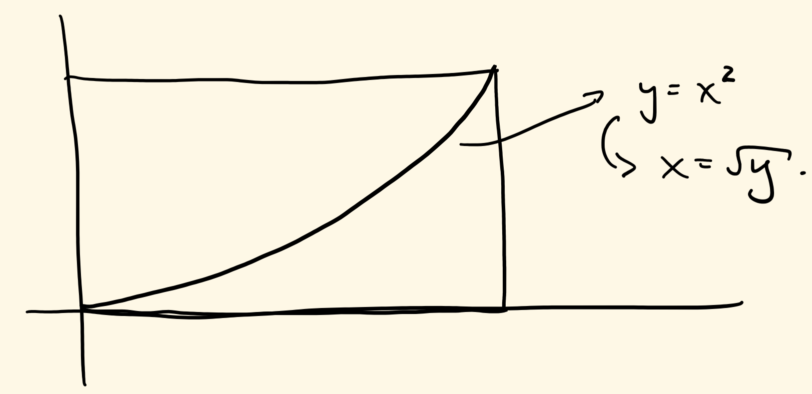
Monday, November 2, 2020 3:53 PM

Ex. A thin plate lying in the xy-plane is bounded by:

$$x^2 + 4y^2 = 12, \quad \rightsquigarrow \quad y = \pm\frac{1}{2}\sqrt{12 - x^2}$$

and

$$x = 4y^2, \quad \rightsquigarrow \quad y = \pm\frac{1}{2}\sqrt{x}$$



with mass density $\rho(x,y) = x$.

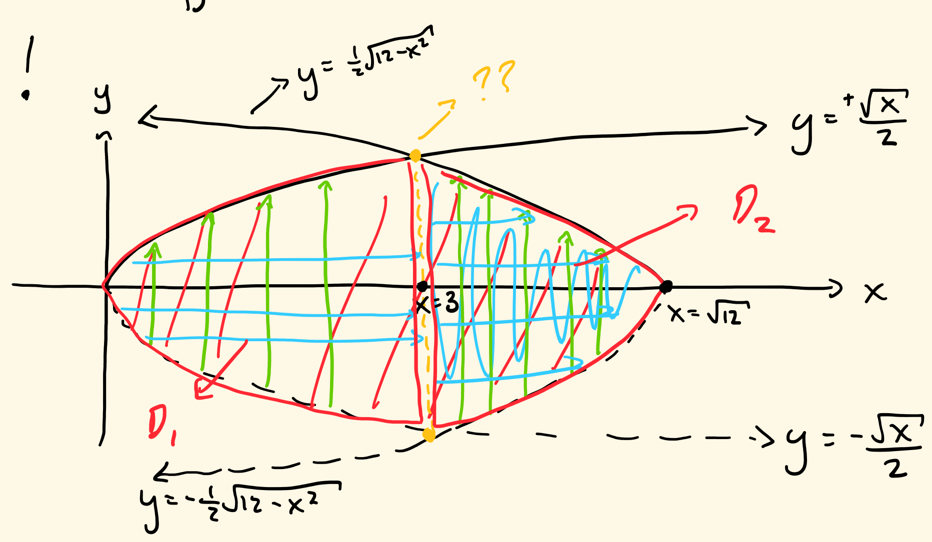
What is the total mass of the plate?

First:

$$m = \iint_D \rho(x,y)\,dx\,dy$$

Draw the region!

So:



D!

So, when does $\frac{\sqrt{x}}{\cancel{2}} = \frac{1}{\cancel{2}}\sqrt{12 - x^2}$?

$$\hookrightarrow x^2 + x - 12 = 0$$

$$\hookrightarrow (x+4)\underbrace{(x-3)}_{x=3} = 0$$

So, $D_1 = \{(x,y) : -\frac{1}{2}\sqrt{x} \le y \le \frac{1}{2}\sqrt{x},\ 0 \le x \le 3\}$

$D_2 = \{(x,y) : -\frac{1}{2}\sqrt{12-x^2} \le y \le \frac{1}{2}\sqrt{12-x^2},\ 3 \le x \le \sqrt{12}\}$

So,

$$m = \iint_D \rho(x,y)\,dy\,dx = \underbrace{\int_0^3 \int_{-\frac{\sqrt{x}}{2}}^{\frac{\sqrt{x}}{2}} x\,dy\,dx}_{\text{I}} + \underbrace{\int_3^{\sqrt{12}} \int_{-\frac{\sqrt{12-x^2}}{2}}^{\frac{\sqrt{12-x^2}}{2}} x\,dy\,dx}_{\text{II}}$$

I: $$\int_0^3 x\left(y \Big|_{y=-\frac{\sqrt{x}}{2}}^{y=\frac{\sqrt{x}}{2}} \right) dx = \int_0^3 x \cdot \sqrt{x}\,dx$$

$$= \int_0^3 x^{3/2}\,dx = \frac{2}{5}x^{\frac{5}{2}}\Big|_{x=0}^{x=3} = \boxed{\frac{2}{5}\cdot 3^{5/2}}$$

II: $$\int_3^{\sqrt{12}} x\left(y \Big|_{-\frac{1}{2}\sqrt{12-x^2}}^{\frac{1}{2}\sqrt{12-x^2}} \right) dx = \int_3^{\sqrt{12}} x \cdot \sqrt{12 - x^2}\,dx$$

Set:

$u = 12 - x^2$

$-\frac{1}{2}du = \cancel{-2}x\,dx$

$$= \int_3^{\sqrt{12}} \left(-\frac{1}{2}\right)\sqrt{u}\,du = \left(-\frac{1}{\cancel{2}}\right)\cdot\left(\frac{\cancel{2}}{3}\right)u^{3/2}\Big|_{x=3}^{x=\sqrt{12}}$$

$$= -\frac{1}{3}\left[(12 - x^2)^{3/2}\Big|_3^{\sqrt{12}}\right] = 0 - \left(-\frac{1}{3}\right)\cdot 3^{3/2}$$

$$= \boxed{3^{1/2}}$$

$$m = \iint_{D_1} \rho + \iint_{D_2} \rho$$

$$= \boxed{\frac{23\sqrt{3}}{5}}$$