EL06 Lab 10 - live

Monday, November 23, 2020 3:55 PM

Topics: - conservative vector fields
- FTOLI

Recall: Given a vector field $F := \langle P, Q \rangle = \langle P(x,y), Q(x,y) \rangle$

We call F conservative if there exists $f(x,y)$:

$$F = \nabla f \rightarrow \text{"potential of } F\text{"}$$

$$\langle P, Q \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = P \\ \frac{\partial f}{\partial y} = Q \end{cases}$$

$F = \langle y^3, -x^3 \rangle$

A quick check: A vector field is conservative if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

From last time: $F = \langle y^3, -x^3 \rangle = \langle P, Q \rangle$

$$\frac{\partial P}{\partial y} = 3y^2, \quad \frac{\partial Q}{\partial x} = -3x^2$$

not equal!

$\Rightarrow$ not conservative!

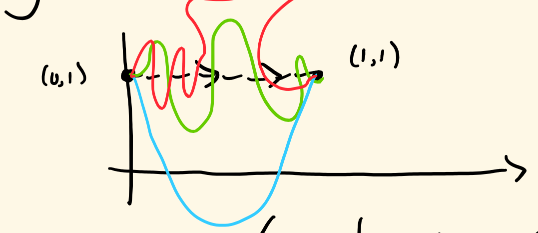
FTOLI: If F is a conservative vector field, then

$$\boxed{\int_C F \cdot dr = f(r(t))\Big|_{t=\alpha}^{t=\beta} = f(r(\beta)) - f(r(\alpha)),}$$

where C is parametrized by $r(t)$ for $\alpha \le t \le \beta$.

Ex. Evaluate $\int_C \arctan(y)\,dx + \frac{x}{1+y^2}\,dy$, where C is any curve from $(0,1)$ to $(1,1)$.

Picture:



Note: $F(x,y) = \langle \arctan(y), x/(1+y^2) \rangle \equiv \langle P, Q \rangle$

check: $\frac{\partial P}{\partial y} = \frac{1}{1+y^2} = \frac{1}{1+y^2} = \frac{\partial Q}{\partial x}$

Conservative!!

We need $f(x,y)$ so that

$$\begin{cases} \frac{\partial f}{\partial x} = \arctan(y) \rightsquigarrow f(x,y) = \boxed{x\arctan(y)} + \cancel{g(y)} \\ \frac{\partial f}{\partial y} = \frac{x}{1+y^2} \rightsquigarrow f(x,y) = \boxed{x\arctan(y)} + \cancel{h(x)} \end{cases}$$

choose $g(y) = h(x) = 0$

So, $f(x,y) = x\arctan(y)$, $\frac{\partial f}{\partial x} = \arctan(y)$, $\frac{\partial f}{\partial y} = \frac{x}{1+y^2}$ ✓

Hence, $f(x,y)$ is our potential for $F(x,y)$.

$\hookrightarrow F = \nabla f$

Then, $r(t)$ (our curve) must satisfy

$r(\alpha) = \langle 0, 1 \rangle$, $r(\beta) = \langle 1, 1 \rangle$

Thus, by the FTOLI:

$$\int_C F \cdot dr = f(r(t))\Big|_{t=\alpha}^{t=\beta}$$

$$= f(r(\beta)) - f(r(\alpha))$$

$$= x\arctan(y)\Big|_{(x,y)=(1,1)} - x\arctan(y)\Big|_{(x,y)=(0,0)}$$

$$= 1 \cdot \arctan(1) - 0 \cdot \arctan(1)$$

$$= \boxed{\arctan(1)}$$