Conservative fields

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Suppose we have a vector field $F = \langle P, Q \rangle$.

Why does $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \implies F$ is conservative?

Well, we want a function f so that $F = \nabla f$.

Suppose $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Then, set

$$f = \int^x P(s,y)\,ds + \int^y Q(x,s)\,ds .$$

Then,

$$\frac{\partial f}{\partial x} = P(x,y) + \int^y \frac{\partial Q}{\partial x}(x,s)\,ds$$

$$\frac{\partial f}{\partial y} = Q(x,y) + \int^x \frac{\partial P}{\partial y}(s,y)\,ds .$$

This isn't quite what we want. But, we know that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. Can we adjust our choice f so that the integrals we don't want disappear?

Set

$$\tilde{f} = \int^x P(s,y)\,ds + \int^y Q(x,s)\,ds - \int^x\!\!\int^y \frac{\partial Q}{\partial s}(s,t)\,dt\,ds$$

Then,

$$\frac{\partial \tilde{f}}{\partial x} = P(x,y) + \int^y \frac{\partial Q}{\partial x}(x,s)\,ds - \int^y \frac{\partial Q}{\partial x}(x,t)\,dt$$

$$= P(x,y)$$

and

$$\frac{\partial \tilde{f}}{\partial y} = Q(x,y) + \int^x \frac{\partial P}{\partial y}(s,y)\,ds - \int^x \frac{\partial Q}{\partial s}(s,y)\,ds .$$

But we know that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$! (Note that whether we call it $\frac{\partial Q}{\partial x}$ or $\frac{\partial Q}{\partial s}$ doesn't matter. We only care about the "first" variable vs. the "second" variable).

Thus,

$$\frac{\partial \tilde{f}}{\partial y} = Q(x,y).$$

Hence, if we assume $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, we have that $F = \langle P, Q \rangle$ is conservative,

AND its potential $f(x,y)$ is given by:

$$f(x,y) = \int^x P(s,y)\,ds + \int^y Q(x,s)\,ds - \int^x\!\!\int^y \frac{\partial Q}{\partial s}(s,t)\,dt\,ds$$

$$= \int^x P(s,y)\,ds + \int^y Q(x,s)\,ds - \int^x\!\!\int^y \frac{\partial P}{\partial s}(t,s)\,ds\,dt ,$$