Lab 5

Monday, May 18, 2020 12:36 PM

Topics: - maximum/minimum problems.

Def^{n}: Given a function $z = f(x,y)$, a point (a,b) is said to be a *local* (or *relative*) *maximum* if $f(a,b) \geq f(x,y)$ for all (x,y) near (a,b).

Similarly, if $f(a,b) \leq f(x,y)$ for all (x,y) near (a,b), (a,b) is said to be a *local* (or *relative*) *minimum*.

How do we find these points? Similar to 1-D, we find the *critical points* of $f(x,y)$. Critical points come in two forms:

(i) $f_x(a,b) = f_y(a,b) = 0$, or

(ii) Either f_x or f_y D.N.E. at the point (a,b).

From this, we see that every relative min/max is a critical point, but not all critical points are relative max/min.

Second Derivative Test: Define the quantity

$$D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}^2(a,b)$$

wherever f_{xx}, f_{xy}, f_{yy} exist. Then, if

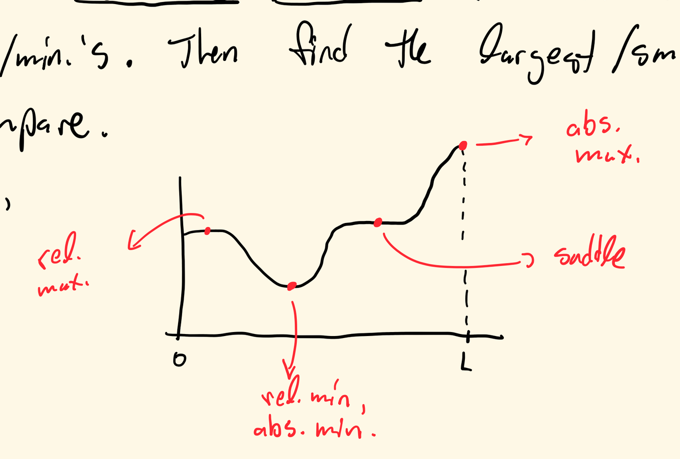
(i) $D(a,b) > 0$ *and* $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local min

(ii) $D(a,b) > 0$ *and* $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local max

(iii) $D(a,b) < 0$ then (a,b) is not a rel. max. or rel. min. (a,b) is called a *saddle* in this case.

So, to find the *absolute maximum* (or minimum), first find all relative max./min.'s. Then find the largest/smallest boundary values and compare.

Recall: In 1-D,



Let's do some examples!

Ex. Find and classify all critical points of

$$f(x,y) = x^4 - 12xy + y^4$$

Since f is a polynomial, we know all derivatives exist and are continuous. (Hence, $f_{xy} = f_{yx}$).

First, find derivatives:

$$f_x = 4x^3 - 12y, \quad f_{xx} = 12x^2$$
$$f_{xy} = -12 = f_{yx}$$
$$f_y = -12x + 4y^3, \quad f_{yy} = 12y^2$$

So, critical points are all of class (i), since all derivatives exist & are continuous. Thus, look for (x,y) such that

$$0 = 4x^3 - 12y \quad \text{and}$$
$$0 = 4y^3 - 12x$$

Eq^{n} II $\rightsquigarrow$ $12x = 4y^3$, rearrange

$$x = \left(\frac{y^3}{3}\right), \quad \text{cube}$$
$$x^3 = \left(\frac{y^3}{3}\right)^3. \quad \text{Sub into I:}$$

$$12y = 4x^3 = 4\left(\frac{y^3}{3}\right)^3 = \frac{4}{27}y^9$$

So,

$$0 = y\left(12 - \frac{4}{27}y^8\right), \quad \text{or}$$
$$0 = \frac{4y}{27}\left(\frac{12 \cdot 27}{4} - y^8\right), \quad \text{so}$$

$y = 0$, or $y = \sqrt[8]{\frac{12 \cdot 27}{4}} = \sqrt[8]{3 \cdot 27} = \sqrt[8]{81} = \sqrt[8]{3^4} = 3^{4/8} = \sqrt{3}$.

Then, solve for x! If $y = 0$, $x = 0$.

If $y = \sqrt{3}$,

$$0 = 4x^3 - 12y = 4x^3 - 12\sqrt{3}$$
$$\hookrightarrow x^3 = \frac{12\sqrt{3}}{4} = 3\sqrt{3}$$
$$\hookrightarrow x = \left(3^{\frac{3}{2}}\right)^{1/3} = \pm\sqrt{3}.$$

So, critical points are $(a,b) = (0,0)$ and $(a,b) = (\sqrt{3},\sqrt{3}), (-\sqrt{3},-\sqrt{3})$

Check the first: $D(0,0) = 0 - (-12)^2 = -144 < 0$.

and $f_{xx}(0,0) = 0$.

Similarly, $D(\sqrt{3},\sqrt{3}) = (12\sqrt{3}^2)(12\sqrt{3}^2) - (-12)^2 = 12^2 \cdot 3^2 - 12^2 > 0$ (Same for $(-\sqrt{3},-\sqrt{3})$)

and $f_{xx}(\sqrt{3},\sqrt{3}) = 12(\sqrt{3})^2 > 0$.

So, our table looks like:

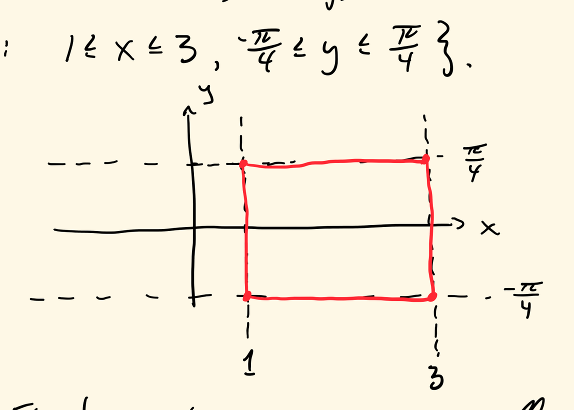
Critical Point	D	f_{xx}	Class
$(0,0)$	< 0	$= 0$	saddle
$\pm(\sqrt{3},\sqrt{3})$	> 0	> 0	loc. min.

Ex. Find the abs. min & max of

$$f(x,y) = (x^2 - 4x)\cos(y)$$

on $\{(x,y) : 1 \leq x \leq 3, -\frac{\pi}{4} \leq y \leq \frac{\pi}{4}\}$.

So, we have



Critical points: Function is nice, so all derivatives exist & are continuous. Let's find derivatives:

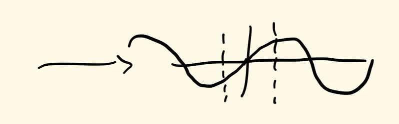
$$f_x = (2x-4)\cos(y), \quad f_{xx} = 2\cos(y)$$
$$f_{xy} = -(2x-4)\sin(y),$$
$$f_y = -(x^2-4x)\sin(y), \quad f_{yy} = -(x^2-4x)\cos(y)$$

So,

$$\begin{cases} 0 = f_x = (2x-4)\cos(y) \\ 0 = f_y = -(x^2-4x)\sin(y) = -x(x-4)\sin(y). \end{cases}$$

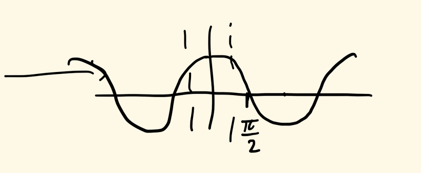
So, if $x = 2$, first eq^{n} is zero.

The second is *not* zero, so $\sin(y) = 0$ must hold. This can only occur at $y = 0$ (since $-\pi/4 \leq y \leq \pi/4$).



So, $(a,b) = (2,0)$

Then, if $x = 0$ or $x = 4$, second eq^{n} is zero. The first eq^{n} is *not* zero in either case, and so $\cos(y) = 0$ is the only option. But $\cos(y) > 0$ for our domain, so we only have one critical point.



Then,

$$D(2,0) = (2\cos(0))\left(-(2^2 - 4 \cdot 2)\cos(0)\right) - \left(-(2(2)-4)\sin(0)\right)^2 = 2(4) \cdot (1) = 8 > 0$$

$$f_{xx}(2,0) = 2\cos(0) > 0$$

So, $(2,0)$ is a local minimum.

Now we check the boundaries. We must check all 4, and we can do so by setting x or y in the following way:

(1) $f(1,y)$ (2) $f(3,y)$

(3) $f(x,-\frac{\pi}{4})$ (4) $f(x,\frac{\pi}{4})$.

So, (1) $f(1,y) = -3\cos(y)$

$$\frac{\partial f}{\partial y} = 3\sin(y) = 0 \quad \text{when } y = 0.$$

So, critical pt. of $f(1,y)$ is $y = 0$, and $f(1,0) = -3$.

(2) $f(3,y) = (3^2 - 4 \cdot 3)\cos(y) = -3\cos(y)$ $\rightsquigarrow$ same as (1), so $f(3,0) = -3$.

(3) & (4) $f(x,-\frac{\pi}{4}) = (x^2-4x)\cos(-\frac{\pi}{4}) = (x^2-4x)\cos(\frac{\pi}{4}) = f(x,\frac{\pi}{4})$, since cos is *even*!

So, $\frac{\partial f}{\partial x}(x,-\frac{\pi}{4}) = \frac{\sqrt{2}}{2}(2x-4) = 0$ when $x = 2$.

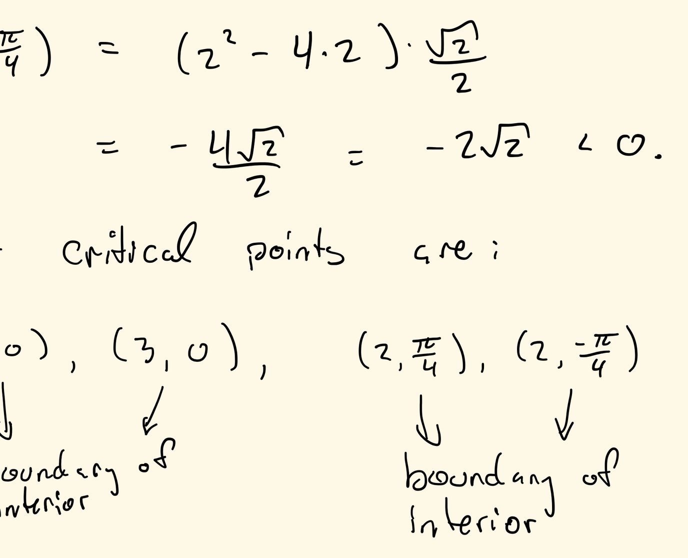
So, $f(2,-\frac{\pi}{4}) = (2^2 - 4 \cdot 2) \cdot \frac{\sqrt{2}}{2} = -\frac{4\sqrt{2}}{2} = -2\sqrt{2} < 0$.

So, all our critical points are:

$(2,0)$, $(1,0)$, $(3,0)$, $(2,\frac{\pi}{4})$, $(2,-\frac{\pi}{4})$, + four corners.

$(2,0)$ → interior; $(1,0)$, $(3,0)$ → boundary of interior; $(2,\frac{\pi}{4})$, $(2,-\frac{\pi}{4})$ → boundary of interior

Visually:



$\sqrt{2}2 > \frac{3}{2}\sqrt{2}$

$-\frac{3}{2}\sqrt{2} > -2\sqrt{2}$

So,

$$f(2,0) = -4$$
$$f(2,\tfrac{\pi}{4}) = f(2,-\tfrac{\pi}{4}) = -2\sqrt{2}$$
$$f(1,\tfrac{\pi}{4}) = f(1,-\tfrac{\pi}{4}) = -3/\sqrt{2}$$
$$f(3,\tfrac{\pi}{4}) = f(3,-\tfrac{\pi}{4}) = -3/\sqrt{2}$$
$$f(1,0) = -3$$
$$f(3,0) = -3.$$

$-2\sqrt{2}$

$-\frac{3}{\sqrt{2}} = -\frac{3}{2}\sqrt{2}$

Hence, the absolute maximum is $-3/\sqrt{2}$ and the absolute minimum is -4.