EL18 Lab 7 - live

Tuesday, October 20, 2020 5:38 PM

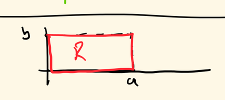
Topics: - double integrals
- regular stuff
- changing the order
- polar coords
- centre of mass (application!)

Ex. $$\int_0^3\int_{-2}^5 (2xy+x)\,dx\,dy = \int_0^3\left(\underbrace{\int_{-2}^5 (2xy+x)\,dx}_{\text{in } x}\right)dy$$

$$\int_{-2}^5 (2xy+x)\,dx = \left(x^2y + \frac{x^2}{2}\right)\Bigg|_{x=-2}^{x=5} = \left(5^2y + \frac{5^2}{2} - \left[(-2)^2y + \frac{(-2)^2}{2}\right]\right)$$
$$= \boxed{21y + \frac{21}{2}}$$

$$\int_0^3 \left[21y + \frac{21}{2}\right]dy = \left(\frac{21}{2}y^2 + \frac{21}{2}y\right)\Bigg|_0^3$$
$$= \frac{21}{2}(3)^2 + \frac{21}{2}\cdot(3) = \frac{21}{2}(3^2+3) = \boxed{126}$$

Ex. $\iint_R x^2y^2\,dA$, $R := [0,a]\times[0,b]$ $\longrightarrow$



$$\iint_R x^2y^2\,dA = \int_0^b\left(\int_0^a x^2y^2\,dx\right)dy$$
$$= \int_0^b\left[\frac{x^3}{3}\cdot y^2\Big|_0^a\right]dy$$
$$= \int_0^b \frac{a^3}{3}y^2\,dy = \frac{a^3}{3}\cdot\frac{y^3}{3}\Bigg|_0^b = \boxed{\frac{a^3\cdot b^3}{9}}$$

Ex. $$\int_0^1\int_0^x xy\,dy\,dx = \int_0^1\left(x\int_0^x y\,dy\right)dx \longrightarrow \int_0^x y\,dy = \frac{y^2}{2}\Bigg|_0^x = \frac{x^2}{2}$$
$$= \int_0^1 x\cdot\frac{x^2}{2}dx = \frac{1}{2}\int_0^1 x^3\,dx = \frac{1}{2}\cdot\frac{1}{4}x^4\Big|_0^1$$
$$= \frac{1}{8}\cdot 1^4 = \boxed{\frac{1}{8}}$$

Ex. $\iint_D e^{y^2}\,dA$, $D = \{(x,y): 0 \le x \le y,\ 0 \le y \le 1\}$.

$$\int_0^1\left(\int_0^y e^{y^2}\,dx\right)dy = \int_0^1\left(e^{y^2}\int_0^y dx\right)dy$$

does not depend on x!

$$= \int_0^1 e^{y^2}\cdot\left(x\Big|_0^y\right)dy = \int_0^1 y e^{y^2}\,dy.$$

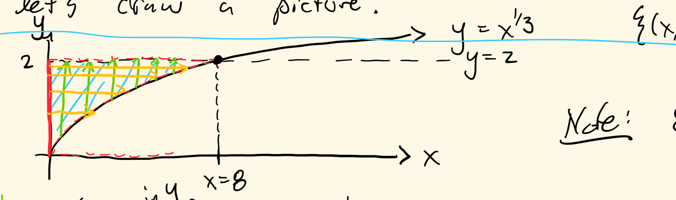
set $u = y^2$, $\frac{du}{2} = y\,dy$

$$= \frac{1}{2}\int_0^1 e^u\,du = \frac{1}{2}e^u\Big|_0^1 = \frac{1}{2}e^{y^2}\Big|_0^1$$
$$= \frac{1}{2}(e-1) = \boxed{\frac{e-1}{2}}$$

Ex. $$\int_0^8\int_{x^{1/3}}^2 \frac{1}{y^4+1}\,dy\,dx \neq \int_{x^{1/3}}^2\int_0^8 \frac{1}{y^4+1}\,dx\,dy$$

$\int \frac{1}{x^2+1}$

So, let's draw a picture!



$\{(x,y): x^{1/3} \le y \le 2,\ 0 \le x \le 8\}$

Note: $8^{1/3} = 2$!

First: integrate in y from $y = x^{1/3}$ to $y = 2$

Second: integrate in x from $x = 0$ to $x = 8$

Alternatively: we can integrate in x from $0 \to y^3$ (if $y = x^{1/3} \Rightarrow x = y^3$)

then, integrate in y from $0 \to 2$

That is, $D = \{(x,y): x^{1/3} \le y \le 2,\ 0 \le x \le 8\}$ (*)
$= \{(x,y): 0 \le x \le y^3,\ 0 \le y \le 2\}$

Then, $$\int_0^8\int_{x^{1/3}}^2 \frac{1}{y^4+1}\,dy\,dx = \int_0^2\left(\int_0^{y^3}\frac{1}{y^4+1}\,dx\right)dy$$

indep. of x !!

$$= \int_0^2 \frac{1}{y^4+1}\left(\int_0^{y^3}dx\right)dy = \int_0^2 \frac{1}{y^4+1}\,y^3\,dy$$

$\longrightarrow$ $u = y^4+1$, $\frac{du}{4} = y^3\,dy$

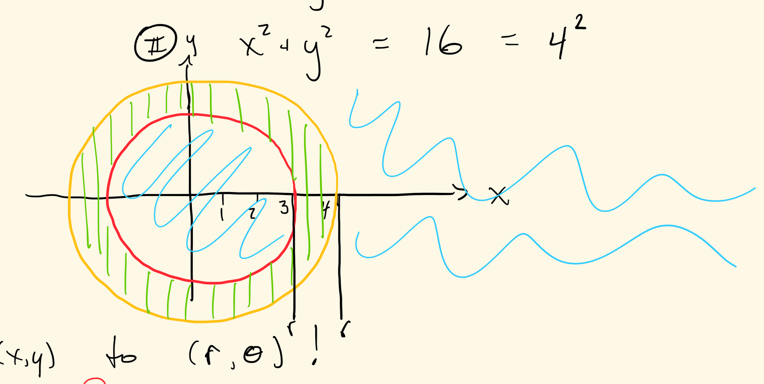
$$= \frac{1}{4}\int_0^2 \frac{1}{u}\,du = \frac{1}{4}\ln(u)\Big|_{y=0}^{y=2}$$
$$= \frac{1}{4}\ln(y^4+1)\Big|_0^2 = \frac{1}{4}\left(\ln(2^4+1) - \ln(1)\right)$$
$$= \boxed{\frac{\ln(17)}{4}}$$

Ex. $\iint_R \sqrt{x^2+y^2}\,dA$, R is the region between the curves

(I) $x^2+y^2 = 9 = 3^2$

(II) $x^2+y^2 = 16 = 4^2$

So, draw R:



So, change from (x,y) to (r,θ)!

First: $0 \le \theta \le 2\pi$

Second: $3 \le r \le 4$.

So, $R = \{(r,\theta): 0 \le \theta \le 2\pi,\ 3 \le r \le 4\}$

Set $x = r\cos\theta$, $y = r\sin\theta$ } $x^2+y^2 = r^2$, so $\sqrt{x^2+y^2} = r$ $(r > 0 \Rightarrow |r| = r)$

Also: $\boxed{dx\,dy = r\,dr\,d\theta}$

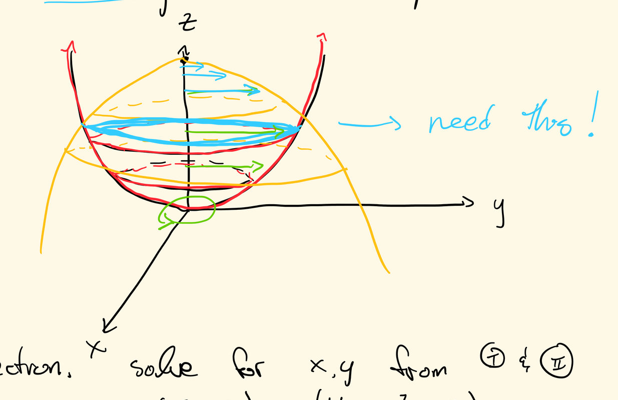
$$\iint_R \sqrt{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_3^4 r\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_3^4 r^2\,dr\,d\theta$$
$$= \int_0^{2\pi}\frac{r^3}{3}\Bigg|_3^4 d\theta = 2\pi\left(\frac{4^3}{3} - \frac{3^3}{3}\right)$$
$$= \boxed{\frac{74\pi}{3}}$$

Ex. Find the volume lying between the curves:

(I) $z = x^2+y^2$ $\rightsquigarrow$ opens up!

(II) $3z = 4 - x^2 - y^2$ $\rightsquigarrow$ opens down!

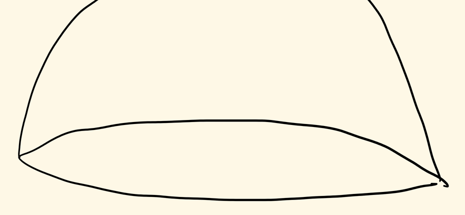
Draw a picture!



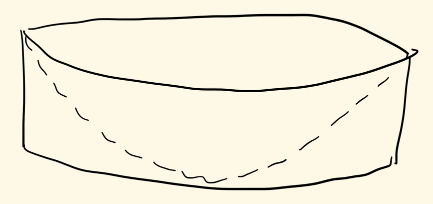
need this!

whole thing! (upper hemisphere piece)

1st piece:



2nd piece:



under the curve! (a bowl)

So, we want the top piece, but without the bowl (but leave the inside of the bowl!)

So, to find the intersection, solve for x,y from (I) & (II)

$$3(\mathrm{I}) - (\mathrm{II}) = 3z - 3z = 3(x^2+y^2) - (4 - x^2 - y^2)$$
$$0 = 4(\underbrace{x^2+y^2-1}_{=0})$$

disk of radius 1

The curves intersect along the circle $x^2+y^2 = 1$

$D = \{(r,\theta): 0 \le \theta \le 2\pi,\ 0 \le r \le 1\}$

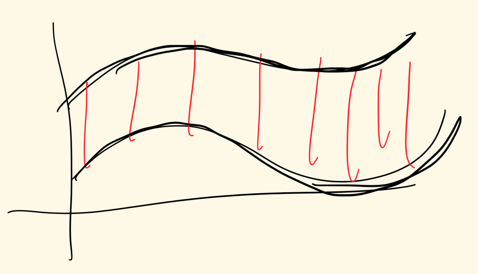
where $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

Next, write $f(x,y) = z$

Upper: (II) $z = \frac{4}{3} - \left(\frac{x^2+y^2}{3}\right) = \frac{4-r^2}{3}$

Lower: (I) $z = x^2+y^2 = r^2$

So, $$\iint f(x,y)\,dx\,dy = \int_0^{2\pi}\int_0^1\left(\left(\frac{4-r^2}{3}\right) - r^2\right)r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1\left(\frac{4}{3} - \frac{4r^2}{3}\right)r\,dr\,d\theta = \frac{4}{3}\cdot 2\pi\int_0^1 r^3\,dr$$
$$= \frac{4}{3}2\pi\left(\frac{r^4}{4}\Big|_0^1\right)$$
$$= \frac{\cancel{4}}{3}2\pi\cdot\frac{1}{\cancel{4}}$$
$$= \boxed{\frac{2\pi}{3}}$$



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