EL18 Lab 5 - live

Tuesday, October 6, 2020 5:57 PM

- max/min problems.

Second Derivative Test: Define the quantity $D(a,b)$ by:

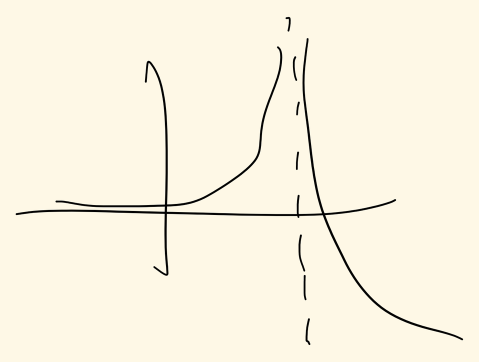
$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)$$

(i) $D(a,b) > 0$ and $f_{xx}(a,b) > 0 \Rightarrow f(a,b)$ is a local min !

(ii) $D(a,b) > 0$ and $f_{xx}(a,b) < 0 \Rightarrow f(a,b)$ is a local max !

(iii) $D(a,b) < 0 \Rightarrow (a,b)$ is neither max. nor min.
(a,b) is a saddle.

Ex. Find and classify all critical points of
$f(x,y) = x^4 - 12xy + y^4$.



First:

$$f_x = 4x^3 - 12y \longrightarrow f_{xx} = 12x^2$$
$$f_y = -12x + 4y^3 \longrightarrow f_{yy} = 12y^2, \quad f_{xy} = \boxed{-12} = f_{yx}$$

So,

$$\left.\begin{aligned} f_x &= 4x^3 - 12y = 0 \\ f_y &= 4y^3 - 12x = 0 \end{aligned}\right\} \begin{matrix} \text{I} \\ \text{II} \end{matrix} \qquad (0,0).$$

Eq^n II:
$$12x = 4y^3$$
$$\hookrightarrow x = \frac{4}{12}y^3 = \frac{1}{3}y^3$$

cube:
$$x^3 = \left(\frac{y^3}{3}\right)^3 = \frac{y^9}{27}$$

4:
$$4x^3 = \frac{4}{27}y^9.$$

Eq^n I:
$$12y = 4x^3 = \frac{4}{27}y^9$$
$$\hookrightarrow \underbrace{y}_{y=0}\underbrace{\left(12 - \frac{4}{27}y^8\right)}_{=0}$$

$$12 = \frac{4}{27}y^8$$
$$\hookrightarrow y^8 = \frac{12\cdot 27}{4} = 3\cdot 27 = 3^4$$
$$y^8 = 3^4$$
$$\hookrightarrow y = \sqrt[8]{3^4} = 3^{1/2} = \pm\sqrt{3}$$

$y = 0,\ y = \pm\sqrt{3}$.

When $y = +\sqrt{3}$, $12x = 4y^3$
$$x = \frac{1}{3}(\sqrt{3})^3 = \frac{3^{3/2}}{3} = \sqrt{3}$$

When $y = -\sqrt{3}$, $x = -\sqrt{3}$.

So, $f_x = f_y = 0$ when $(x,y) = \begin{cases} (0,0) \\ (\sqrt{3},\sqrt{3}) \\ (-\sqrt{3},-\sqrt{3}) \end{cases}$

Second: Compute $D(a,b)$ for each C.P.

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 0 - [-12]^2 = -144 < 0.$$

$$D(\sqrt{3},\sqrt{3}) = 12(\sqrt{3})^2\cdot 12(\sqrt{3})^2 - 144 = (36)\cdot(36) - 144 = 12^2\cdot 3^2 - 12^2 = 12^2(3^2-1) > 0$$

$$D(-\sqrt{3},-\sqrt{3}) = 12^2(3^2-1) > 0$$

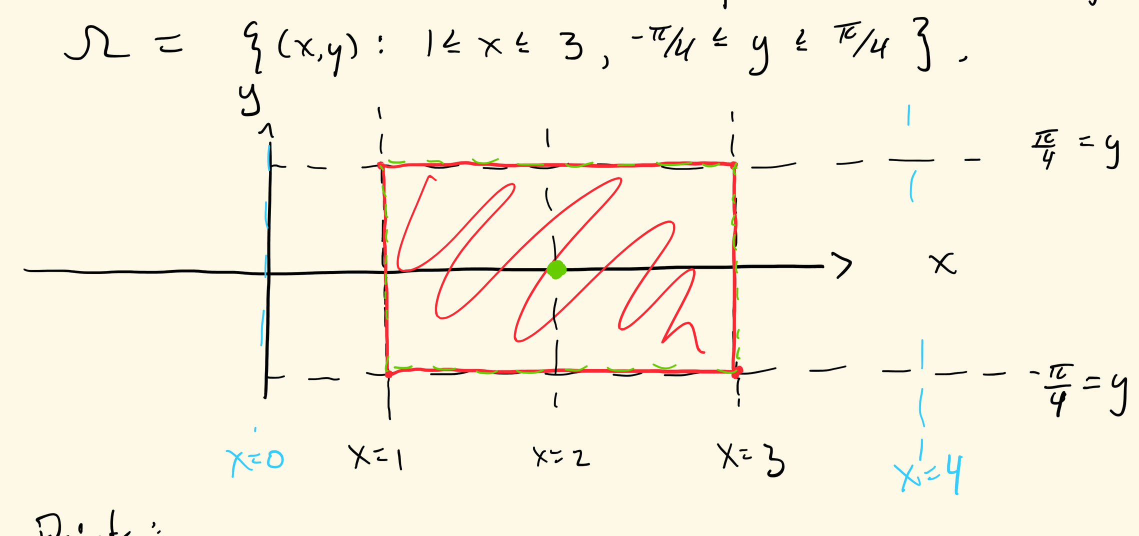
$D(0,0) < 0$, $\boxed{D(\pm\sqrt{3},\pm\sqrt{3}) > 0}$

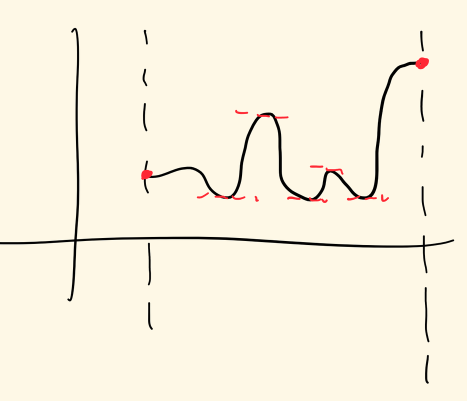
$f_{xx}(0,0) = 0$, $f_{xx}(\sqrt{3},\sqrt{3}) = 12(\sqrt{3})^2 > 0$, $f_{xx}(-\sqrt{3},-\sqrt{3}) = 12(-\sqrt{3})^2 > 0$

Critical Point	$D(a,b)$	f_{xx}	Class
$(0,0)$	< 0	$= 0$	saddle
$\pm(\sqrt{3},\sqrt{3})$	> 0	> 0	local min !

Ex. Find the abs. min & max of $f(x,y) = (x^2-4x)\cos(y)$
on $\Omega = \{(x,y) : 1 \le x \le 3,\ -\pi/4 \le y \le \pi/4\}$.

Ω:



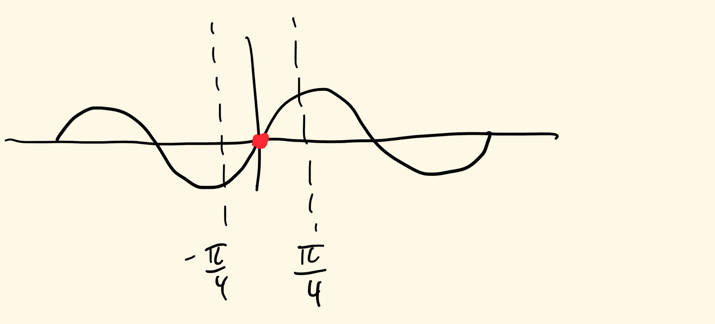


Critical Points:

$$f_x = (2x-4)\cos(y) = 2(x-2)\cos(y) \ ; \quad f_{xx} = 2\cos(y)$$
$$f_y = -(x^2-4x)\sin(y) \qquad f_{yy} = -(x^2-4x)\cos(y)$$
$$\hookrightarrow f_{xy} = -2(x-2)\sin(y). \longrightarrow 0 \ @ \ x = 2$$

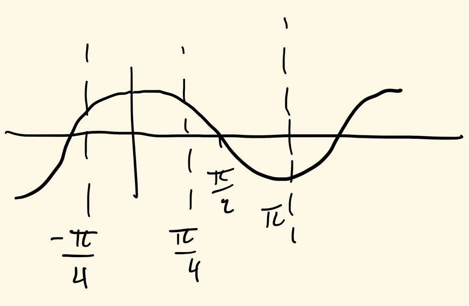
Solve:
$$\begin{cases} 0 = f_x \\ 0 = f_y \end{cases} \Rightarrow \begin{cases} 0 = 2(x-2)\cos(y) \\ 0 = -(x^2-4x)\sin(y) \end{cases}$$

Well, if $x = 2$, $f_x = 0$. What about f_y?
$$f_y(2,y) = -(4 - 4\cdot 2) = \underbrace{4\sin(y)}.$$
how to make this zero? $y = 0$!



So, $\boxed{(a,b) = (2,0)}$ is critical!

Second eq^n: $-(x^2-4x)\sin(y) = -x\underbrace{(x-4)}\sin(y)$.
$\cancel{x=0}$ $\cancel{x=4}$ $\longrightarrow$ Not in the domain!



First eq^n: $\cos(y)$: always > 0 on $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

Next,
$$D(a,b) = D(2,0) = f_{xx}(2,0)f_{yy}(2,0) - [f_{xy}(2,0)]^2$$
$$= 2\cos(0)\cdot(-(2^2 - 4\cdot 2))\cos(0) - [0]^2$$
$$= 2\cdot 1(+4)\cdot 1$$
$$= 8 > 0$$

$$f_{xx}(2,0) = 2\cos(0) = 2 > 0.$$

$(a,b) = (2,0)$ is a local min!

Now the boundaries! There are four of them.

(1) $f(1,y)$ (2) $f(3,y)$
(3) $f(x,-\frac{\pi}{4})$ (4) $f(x,\frac{\pi}{4})$

So,
(1) $f(1,y) = -3\cos(y)$.
$$\frac{\partial f}{\partial y} = 3\sin(y) = 0 \Longrightarrow y = 0 \quad \left[\text{on } \left(-\frac{\pi}{4},\frac{\pi}{4}\right)\right]$$

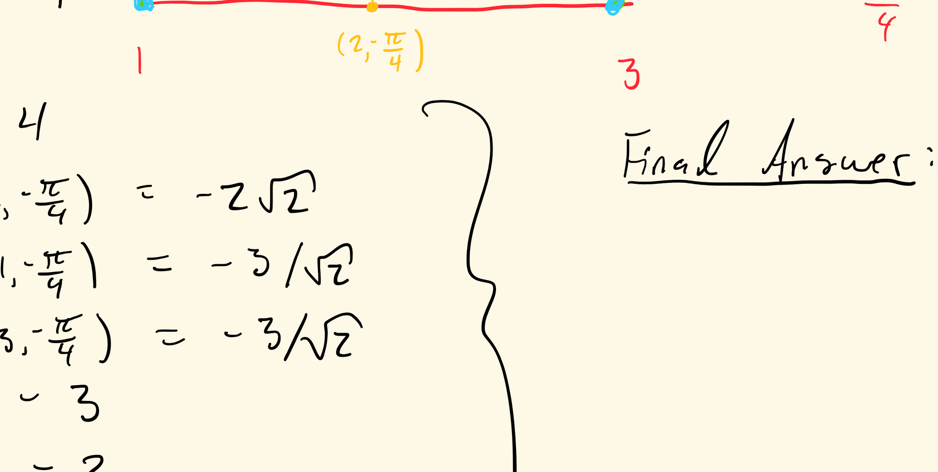
So, C.P. of $f(1,y)$ is $y = 0$.
$f(1,0) = -3$.

(2) $f(3,y) = -3\cos(y) \rightsquigarrow$ same (1)
$$\frac{\partial f}{\partial y}(3,y) = 0 \Rightarrow y = 0.$$
$$f(3,0) = -3$$

(3) + (4)
$$f(x,-\tfrac{\pi}{4}) = (x^2-4x)\cos(-\tfrac{\pi}{4}) \qquad \text{cosine is even!}$$
$$= (x^2-4x)\cos(\tfrac{\pi}{4}).$$
$$\frac{\partial f}{\partial x}(x,\tfrac{\pi}{4}) = (2x-4)\cdot\frac{\sqrt{2}}{2}, \ = 0 \text{ when } x = 2,$$

So,
$$f(2,\tfrac{\pi}{4}) = (2^2 - 4\cdot 2)\frac{\sqrt{2}}{2} = -2\sqrt{2}$$
$$\hookrightarrow f(2,-\tfrac{\pi}{4}) = -2\sqrt{2}.$$

C.P.'s: $(2,0)$ $(1,0)$, $(3,0)$, $(2,\frac{\pi}{4})$, $(2,-\frac{\pi}{4})$
+ four corners,
$(1,\frac{\pi}{4})$, $(3,\frac{\pi}{4})$, $(1,-\frac{\pi}{4})$, $(3,-\frac{\pi}{4})$



$$\left.\begin{aligned} f(2,0) &= -4 \\ f(2,\tfrac{\pi}{4}) &= f(2,-\tfrac{\pi}{4}) = -2\sqrt{2} \\ f(1,\tfrac{\pi}{4}) &= f(1,-\tfrac{\pi}{4}) = -3/\sqrt{2} \\ f(3,\tfrac{\pi}{4}) &= f(3,-\tfrac{\pi}{4}) = -3/\sqrt{2} \\ f(1,0) &= -3 \\ f(3,0) &= -3 \end{aligned}\right\}$$

Final Answer: The abs. maximum is $-3/\sqrt{2}$ @ $(1,\pm\frac{\pi}{4})$, $(3,\pm\frac{\pi}{4})$
The abs. minimum is -4 @ $(2,0)$