EL06 Lab 4 - live

Monday, October 5, 2020 3:49 PM

- Gradient
- Directional derivatives.

Def$^{\underline{n}}$: Given a function $f(x,y) = z$, the gradient of $f(x,y)$ is:

$$\boxed{\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle}.$$

If $f(x_1, x_2, \cdots, x_n)$, then $\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right\rangle$.

Recall: Given $\vec{u} = \langle u_1, \cdots, u_n \rangle$,

$|\vec{u}| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$. (magnitude of $\vec{u}$)

Ex. Find the gradient of $f(x,y) = xy + e^{x-y}$.

So,

$$\frac{\partial f}{\partial x} = y + e^{x-y} \cdot 1 = y + e^{x-y}.$$
$$\frac{\partial f}{\partial y} = x + e^{x-y}(-1) = x - e^{x-y}.$$
$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle y + e^{x-y}, x - e^{x-y} \right\rangle.$$

Something to remember !! The maximum rate of change of a function f is given by the magnitude of the gradient !

$f = f(x,y)$

$$|\nabla f| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

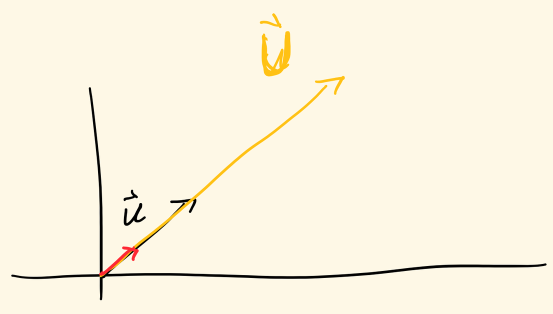
$f = f(x_1, x_2, \cdots, x_n)$, then

$$|\nabla f| = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2}.$$

We can also talk about the rate of change in other directions.

Directional Derivative: Given a vector $\vec{u} = \langle u_1, u_2 \rangle$, the directional derivative of $f(x,y)$ is

$D_{\vec{u}} f = \vec{u} \cdot \nabla f \longrightarrow \langle u_1, u_2 \rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y}$.



If we want $\vec{u}$ to be a unit vector, take

$\vec{v} := \frac{\vec{u}}{|\vec{u}|} \implies |\vec{v}| = 1$.

Ex. Find the directional derivative of $f(x,y) = xy + e^{x-y}$ in the direction $\vec{u} = \langle 1, 2 \rangle$.

We already found ∇f: $\nabla f = \langle y + e^{x-y}, x - e^{x-y} \rangle$

So,

$$D_{\vec{u}} f = \vec{u} \cdot \nabla f = \langle 1, 2 \rangle \cdot \langle y + e^{x-y}, x - e^{x-y} \rangle$$
$$= (y + e^{x-y}) \cdot 1 + (x - e^{x-y}) \cdot 2$$
$$= 2x + y - e^{x-y}.$$

$D_{\vec{u}} f = \,?$

Maximize $(D_{\vec{u}} f)$ over all directions.

biggest when $\vec{u} = \nabla f$.

Normalize $\vec{u}$: $|\vec{u}| = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$.

So, the "unit" directional derivative is

$$\frac{D_{\vec{u}} f}{\sqrt{5}} = \frac{2x + y - e^{x-y}}{\sqrt{5}}.$$

Ex. Find the greatest rate of change of the function $f(x,y) = e^{x-y}$ at $(x_0, y_0) = (1,1)$. In which direction does this occur?

Need ∇f: $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle e^{x-y}, -e^{x-y} \rangle$

So, $\nabla f(1,1)$: $\nabla f(1,1) = \langle e^0, -e^0 \rangle$

$= \langle 1, -1 \rangle$

So, $|\nabla f(1,1)| = \sqrt{(1)^2 + (-1)^2} = \boxed{\sqrt{2}.}$

The greatest rate of change of f @ $(1,1)$ is $\sqrt{2}$. This occurs in the direction $\boxed{\frac{\langle 1, -1 \rangle}{\sqrt{2}}} = \frac{\nabla f(1,1)}{|\nabla f(1,1)|}$

Ex. Find the greatest rate of change of $f(x,y) = \ln(x) y^2 - \sin(y)$ at $(x_0, y_0) = (e, \pi)$.

$$\nabla f = \left\langle \frac{y^2}{x}, 2y \ln(x) - \cos(y) \right\rangle$$

So,

$$\nabla f(e, \pi) = \left\langle \frac{\pi^2}{e}, 2\pi \ln(e) - \cos(\pi) \right\rangle$$
$$= \left\langle \frac{\pi^2}{e}, 2\pi + 1 \right\rangle$$

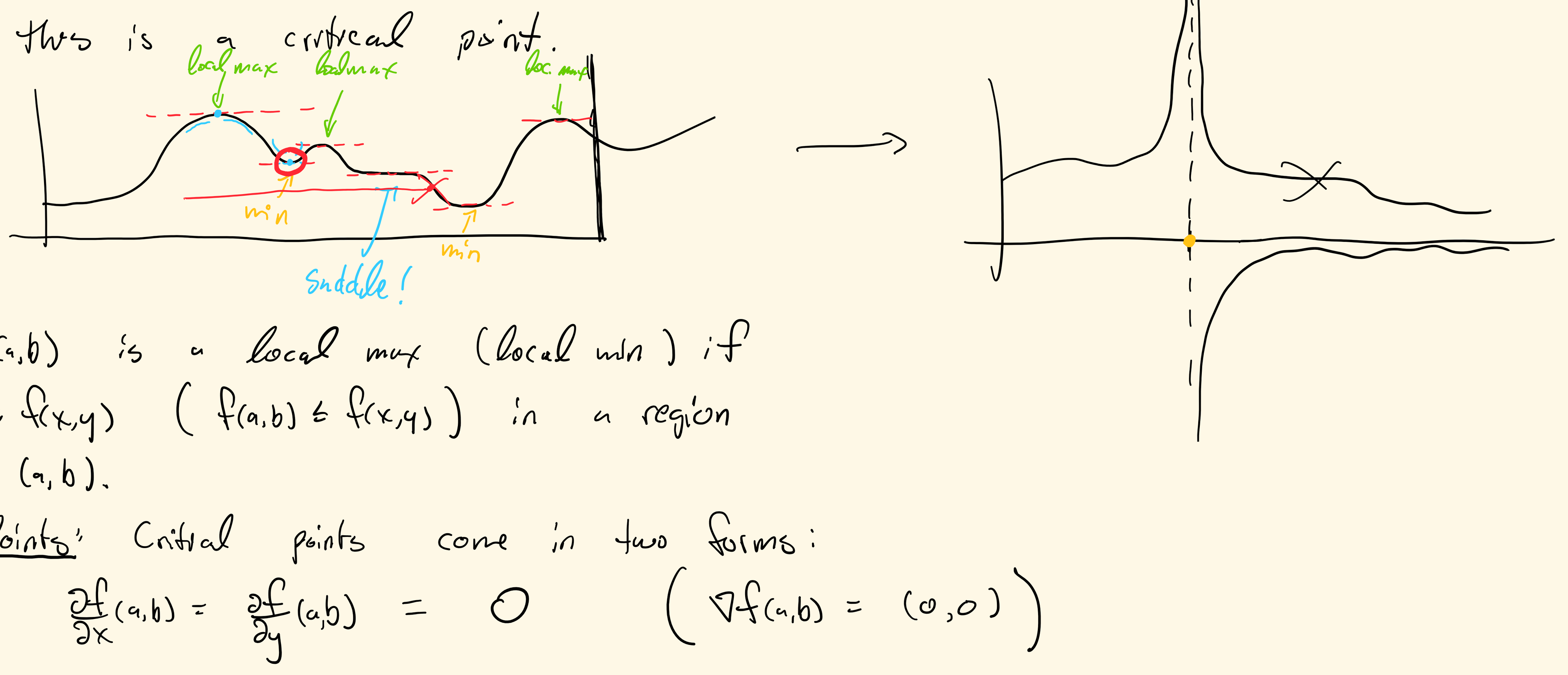
So,

$$|\nabla f(e,\pi)| = \sqrt{\left(\frac{\pi^2}{e}\right)^2 + (2\pi + 1)^2}$$
$$= \sqrt{\frac{\pi^4}{e^2} + (2\pi + 1)^2}.$$

So, the greatest rate of change is: $\sqrt{\frac{\pi^4}{e^2} + (2\pi+1)^2}$

in the direction: $\left\langle \frac{\pi^2}{e}, 2\pi + 1 \right\rangle$.

In 1-D, set $f'(x) = 0$, whatever x satisfies this is a critical point.



A point (a,b) is a local max (local min) if $f(a,b) \geq f(x,y)$ ($f(a,b) \leq f(x,y)$) in a region near (a,b).

Critical Points: Critical points come in two forms:

(i) $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$ $\quad$ ($\nabla f(a,b) = (0,0)$)

(ii) Either $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ D.N.E., in some sense.