Chain Rule Derivation

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$$f = f(x,y), \quad x = x(t), \quad y = y(t).$$

So,

$$\frac{df}{dt} = \lim_{h \to 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t))}{h}$$

Rewrite:

$$\lim_{h \to 0} \frac{f(x(t+h), y(t+h)) \overbrace{- f(x(t), y(t+h)) + f(x(t), y(t+h))}^{=0} - f(x(t), y(t))}{h}.$$

Two pieces:

$$\frac{f(x(t+h), y(t+h)) - f(x(t), y(t+h))}{h} \quad \text{and} \quad \frac{f(x(t), y(t+h)) - f(x(t), y(t))}{h}$$

Now multiply & divide by $x(t+h) - x(t)$ and $y(t+h) - y(t)$:

$$\frac{df}{dt} = \lim_{h \to 0} \left[\frac{f(x(t+h), y(t+h)) - f(x(t), y(t+h))}{x(t+h) - x(t)}\right] \cdot \frac{x(t+h) - x(t)}{h} + \left[\frac{f(x(t), y(t+h)) - f(x(t), y(t))}{y(t+h) - y(t)}\right] \cdot \frac{y(t+h) - y(t)}{h}$$

Now, since we are assuming f is differentiable, the first pieces are:

$$\lim_{h \to 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t+h))}{x(t+h) - x(t)} = \frac{\partial f}{\partial x}.$$

Similarly,

$$\lim_{h \to 0} \frac{f(x(t), y(t+h)) - f(x(t), y(t))}{y(t+h) - y(t)} = \frac{\partial f}{\partial y}$$

For the other pieces, assuming x and y are also differentiable,

$$\lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = \frac{dx}{dt},$$

and

$$\lim_{h \to 0} \frac{y(t+h) - y(t)}{h} = \frac{dy}{dt}.$$

Putting this all together,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$