Lab 9 Live

Tuesday, June 2, 2020 8:07 PM

Topics:
- Triple integrals in spherical coords
- line integrals.

Recall: Spherical coords are given by:

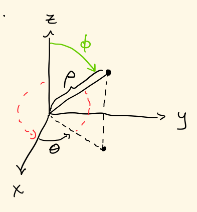
$$\begin{cases} x = \rho\cos\theta\sin\phi \,; \\ y = \rho\sin\theta\sin\phi \,; \\ z = \rho\cos\phi \,, \end{cases} \quad \text{where}$$

$\rho \in [0,\infty)$,

$\theta \in [0,2\pi)$,

$\phi \in [0,\pi)$.

Picture:



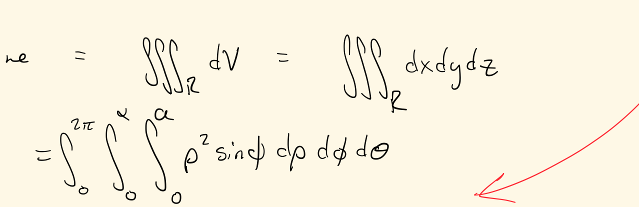
Facts:

$$\rho^2 = x^2+y^2+z^2 .$$

$$dx\,dy\,dz = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta .$$

Ex. Find the vol. of the reg. inside the cone $\phi = \alpha$, and the sphere $\rho = a$.

Picture:



So,
$$R = \begin{cases} 0 \le \rho \le a \\ 0 \le \phi \le \alpha \\ 0 \le \theta \le 2\pi \end{cases}$$

So,
$$\text{Volume} = \iiint_R dV = \iiint_R dx\,dy\,dz$$
$$= \int_0^{2\pi}\int_0^{\alpha}\int_0^{a} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$
$$= \int_0^{2\pi}\int_0^{\alpha} \sin\phi \cdot \frac{a^3}{3}\, d\phi\, d\theta$$
$$= \frac{a^3}{3}\int_0^{2\pi} d\theta \int_0^{\alpha} \sin\phi\, d\phi$$
$$= \frac{a^3}{3}\cdot 2\pi \cdot \left(-\cos\phi\right)\Big|_0^{\alpha} = \boxed{\frac{2\pi a^3}{3}\left(1-\cos\alpha\right)} .$$

Integrate over $0 \le x^2+y^2+z^2 \le 1 \rightsquigarrow 0 \le \rho \le 1$

vs

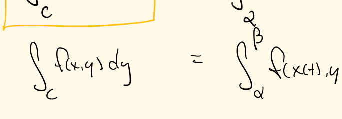
Integrate over $x^2 + \frac{y^2}{36} + \frac{z^2}{16} = 1$

↳ turn into $u^2+v^2+w^2 \le 1$.

set
$u = x \rightsquigarrow dx = du$
$v = \frac{y}{6} \rightsquigarrow dy = 6dv$
$w = \frac{z}{4} \rightsquigarrow dz = 4dw$

↳ $dx\,dy\,dz = 6\cdot 4\, du\,dv\,dw$

Line Integrals: Suppose you're given a curve C in $\mathbb{R}^2$ defined paramet. by
$$r(t) = \langle x(t), y(t)\rangle , \quad \alpha \le t \le \beta .$$



Then, the line integral w.r.t. arclength of a function $f(x,y)$ is:
$$\int_C f(x,y)\,ds = \int_\alpha^\beta f(x(t),y(t))\cdot\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

Also, the line integral of $f(x,y)$ w.r.t. x (or y) is
$$\boxed{\int_C f(x,y)\,dx} = \int_\alpha^\beta f(x(t),y(t))\,\frac{dx}{dt}\cdot dt$$
$$\left(\text{or} \quad \int_C f(x,y)\,dy = \int_\alpha^\beta f(x(t),y(t))\,\frac{dy}{dt}\,dt \right).$$

Finally, the line integral over a vector field $F(x,y) = \langle P(x,y), Q(x,y)\rangle$ along a curve C given by $r(t) = \langle x(t), y(t)\rangle$ for $\alpha \le t \le \beta$,
$$\int_C F\cdot dr = \int_C F\cdot T(s)\,ds = \int_C P\,dx + Q\,dy ,$$
where
$$T(t) = \frac{r'(t)}{|r'(t)|} \longrightarrow \text{the UNIT TANGENT VECTOR along } C.$$

Ex. Evaluate $\int_C F\cdot dr$, where
$$F(x,y) = \frac{\langle x, y\rangle}{(x^2+y^2)^{3/2}} \quad \text{and}$$
$$r(t) = e^t\langle \cos(t), \sin(t)\rangle , \quad \boxed{\text{from } 0 \le t \le 1}$$
$$= \langle x(t), y(t)\rangle$$
$$\implies \begin{aligned} x(t) &= e^t\cos(t) \\ y(t) &= e^t\sin(t). \end{aligned}$$

I'll use $\int_C P\,dx + Q\,dy$. $\left(= \int_C F\cdot dr\right)$

First, what are P & Q ??
$$F(x,y) = \frac{\langle x, y\rangle}{(x^2+y^2)^{3/2}} = \langle P(x,y), Q(x,y)\rangle$$
$$\implies P(x,y) = \frac{x}{(x^2+y^2)^{3/2}}$$
$$Q(x,y) = \frac{y}{(x^2+y^2)^{3/2}}$$
$$x(t) = e^t\cos(t) \rightsquigarrow \frac{dx}{dt} = e^t(\cos(t)-\sin(t))$$
$$y(t) = e^t\sin(t). \rightsquigarrow \frac{dy}{dt} = e^t(\sin(t)+\cos(t))$$

So,
$$\int_C F\cdot dr = \int_C P\,dx + Q\,dy$$
$$= \int_0^1 P(x(t),y(t))\,\frac{dx}{dt}\,dt + \int_0^1 Q(x(t),y(t))\,\frac{dy}{dt}\,dt$$
$$= \int_0^1 \frac{x(t)}{(x(t)^2+y(t)^2)^{3/2}}\cdot\frac{dx}{dt}\,dt + \int_0^1 \frac{y(t)}{(x^2(t)+y^2(t))^{3/2}}\,\frac{dy}{dt}\,dt$$

Note: $x^2+y^2 = e^{2t}(\cos^2(t)+\sin^2(t)) = e^{2t}$

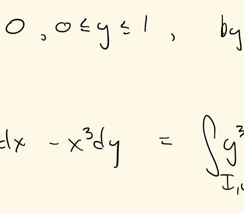
↳ $(x^2+y^2)^{3/2} = e^{2t\cdot\frac{3}{2}} = e^{3t}$.

$$\int_0^1 \frac{e^{t}\cos(t)}{e^{3t}}\cdot\left(e^{t}(\cos(t)-\sin(t))\right)dt + \int_0^1 \frac{e^t\sin(t)}{e^{3t}}\left(e^t(\cos(t)+\sin(t)\right)dt$$

$$= \int_0^1 e^{-t}\cos^2(t)\,dt + \int_0^1 e^{-t}\sin^2(t)\,dt$$
$$= \int_0^1 e^{-t}\underbrace{(\cos^2(t)+\sin^2(t))}_{=1}\,dt$$
$$= \int_0^1 e^{-t}\,dt = -e^{-t}\Big|_0^1 = 1-\frac{1}{e}$$
$$= \boxed{\frac{e-1}{e}}$$

Ex. Evaluate $\int_C y^3dx - x^3dy$, where C is the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$, oriented counterclockwise!

Picture:



Use $\int_C = \int_{I_1} + \int_{I_2} + \int_{I_3}$.

I_1: $y=0$, $0 \le x \le 1$, so $I_1 = \langle t, 0\rangle$ for $0 \le t \le 1$

I_2: $y = 1-x$, $0 \le x \le 1$, so ~~$I_2 = \langle t, 1-t\rangle$ for $0 \le t \le 1$.~~

orientation is wrong?

$I_2 = \langle 1-t, t\rangle$, for $0 \le t \le 1$

I_3: $x=0$, $0 \le y \le 1$, by moving down!

$I_3 = \langle 0, 1-t\rangle$, for $0 \le t \le 1$

So,
$$\int_C y^3dx - x^3dy = \int_{I_1\cup I_2\cup I_3} y^3dx - x^3dy = \int (y(t))^3\frac{dy}{dt}\,dt - (x(t))^3\frac{dx}{dt}\,dt$$

So,
$$\int_{I_1}\left((y(t))^3\frac{dx}{dt} - (x(t))^3\frac{dy}{dt}\right)dt = \int_0^1 0^3\cdot 1 - (t)^3\cdot(0) = 0.$$
$$\int_{I_2}\left((y)^3\frac{dx}{dt} - (x)^3\frac{dy}{dt}\right)dt = \int_0^1\left(t^3\cdot(-1) - (1-t)^3\cdot(1)\right)dt$$
$$= -\frac{t^4}{4}\Big|_0^1 + \frac{(1-t)^4}{4}\Big|_0^1$$
$$= -\frac{1}{4} - \frac{1}{4} = \boxed{-\frac{1}{2}}$$
$$\int_{I_3}\left(y^3\frac{dx}{dt} - x^3\frac{dy}{dt}\right)dt = \int_0^1\left((1-t)^3\cdot 0 - 0\cdot\frac{dy}{dt}\right)dt = 0$$

$$\int_C y^3\frac{dx}{dt} - x^3\frac{dy}{dt} = \int_{I_1} + \int_{I_2} + \int_{I_3}$$
$$= 0 - \frac{1}{2} + 0$$
$$= \boxed{-\frac{1}{2}}$$

Question: Does there exist a field $F = F(x,y)$ such that $\nabla F = \langle y^3, -x^3\rangle$