Lab 9

Monday, June 1, 2020 11:48 AM

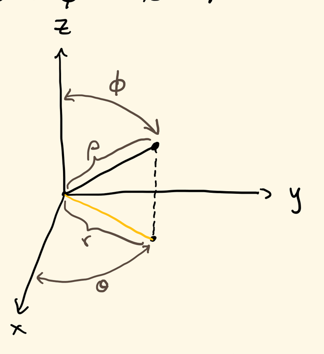
Topics:
- Triple integrals in spherical coords
- Line integrals

Recall: spherical coordinates are given by the following transform:

$$\begin{cases} x = \rho\cos\theta\sin\phi , \\ y = \rho\sin\theta\sin\phi , \\ z = \rho\cos\phi , \end{cases}$$

where
$$0 \le \rho < \infty, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$$

Visually:

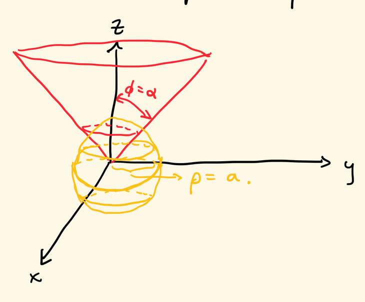


Think about this: why is $\theta \in [0, 2\pi]$ while $\phi \in [0, \pi]$? Why not $\phi \in [0, 2\pi]$?

So, similar to polar coordinates where $x^2 + y^2 = r^2$, spherical coords. has $\boxed{x^2 + y^2 + z^2 = \rho^2}$.
Also, similar to polar coords. where $dxdy = r\,dr\,d\theta$, in spherical coords. we have $\boxed{dxdydz = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$

Ex. Find the volume of the region that lies inside the cone $\phi = \alpha$ and the sphere $\rho = a$.

Picture:



So,
$$\begin{cases} 0 \le \rho \le a, \\ 0 \le \phi \le \alpha, \\ 0 \le \theta \le 2\pi, \end{cases} \quad \text{or}$$

$$R = \{ (r, \phi, \theta) : 0 \le \theta \le 2\pi, \ 0 \le \phi \le \alpha, \ 0 \le \rho \le a. \}$$

Then,
$$\begin{aligned}
\text{Vol.} &= \iiint_R dxdydz \\
&= \int_0^{2\pi}\int_0^{\alpha}\int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi}\int_0^{\alpha} \left(\frac{\rho^3}{3}\sin\phi \Big|_{\rho=0}^{a} \right) d\phi \, d\theta \\
&= \int_0^{2\pi}\int_0^{\alpha} \frac{a^3}{3}\sin\phi \, d\phi \, d\theta \\
&= \int_0^{2\pi} \frac{a^3}{3}\left(-\cos\phi\right)\Big|_{\phi=0}^{\phi=\alpha} d\theta \\
&= \frac{a^3}{3}(1-\cos\alpha)\int_0^{2\pi} d\theta \\
&= \boxed{\frac{2\pi a^3}{3}(1-\cos\alpha)}
\end{aligned}$$

Ex. Compute $\iiint_R z\sqrt{x^2+y^2+z^2}\, dV$, where R is as in the previous example.

So, note $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$, and
$\rho = \sqrt{x^2+y^2+z^2}$, and
$z = \rho\cos\phi$

Then,
$$\begin{aligned}
\iiint_R z\sqrt{x^2+y^2+z^2}\, dV &= \int_0^{2\pi}\int_0^{\alpha}\int_0^{a} \rho\cos\phi \cdot \rho \cdot \rho^2\sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi}\int_0^{\alpha}\int_0^{a} \rho^4 \cos\phi\sin\phi \, d\rho \, d\phi \, d\theta \\
&= 2\pi \cdot \left(\frac{\rho^5}{5}\right)\Big|_{\rho=0}^{\rho=a} \int_0^{\alpha} \cos\phi\sin\phi \, d\phi, \quad \text{set } \begin{matrix} u = \sin\phi \\ du = \cos\phi \, d\phi \end{matrix} \\
&= \frac{2\pi a^5}{5} \cdot \int_0^{\alpha} u \, du \\
&= \frac{2\pi a^5}{5} \cdot \frac{u^2}{2}\Big|_{\phi=0}^{\phi=\alpha} \\
&= \frac{\cancel{2}\pi a^5}{5}\left(\frac{\sin^2\alpha}{\cancel{2}}\right) \\
&= \boxed{\frac{\pi a^5 \sin^2\alpha}{5}}
\end{aligned}$$

Suppose you're given a curve C in $\mathbb{R}^2$ defined parametrically by
$$r(t) = \langle x(t), y(t) \rangle, \quad \alpha \le t \le \beta.$$
Then, the line integral w.r.t. arclength of a function $f(x,y)$ over C is
$$\int_C f(x,y)\,ds = \int_\alpha^\beta f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Similarly, the line integral of $f(x,y)$ over a curve C w.r.t. to x or y is given by
$$\int_C f(x,y)\,dx = \int_\alpha^\beta f(x(t),y(t))\frac{dx}{dt}\,dt$$
$$\int_C f(x,y)\,dy = \int_\alpha^\beta f(x(t),y(t))\frac{dy}{dt}\,dt$$

Finally, the line integral of the vector field $F(x,y) = \langle P(x,y), Q(x,y) \rangle$ along a curve C given by $r(t) = (x(t), y(t))$ for $\alpha \le t \le \beta$ is
$$\int_C F\cdot dr = \int_C F \cdot T(s)\,ds = \int_C (P\,dx + Q\,dy),$$
where $T(t) = \dfrac{r'(t)}{|r'(t)|}$, the unit tangent vector along C at t.

Ex. Evaluate $\int_C F\cdot dr$, where $F(x,y) = \dfrac{\langle x, y\rangle}{(x^2+y^2)^{3/2}}$ and
$$r(t) = e^t\langle \cos t, \sin t\rangle, \text{ for } 0 \le t \le 1.$$
$$\left(= \langle x(t), y(t) \rangle \right)$$

So, $P = \dfrac{x}{(x^2+y^2)^{3/2}}$,

$Q = \dfrac{y}{(x^2+y^2)^{3/2}}$, $\quad r(t) = \left\langle e^t\cos(t), e^t\sin(t) \right\rangle$,

So,
$$\begin{aligned}
r'(t) &= \left\langle e^t\cos t - e^t\sin t, \ e^t\sin t + e^t\cos t \right\rangle \\
&= e^t\langle \cos t - \sin t, \cos t + \sin t\rangle.
\end{aligned}$$

Finally,
$$\begin{aligned}
\int_C F\cdot dr &= \int_C F\cdot T(s)\,ds \\
&= \int_C (P\,dx + Q\,dy) \\
&= \int_0^1 P(x(t),y(t))\frac{dx}{dt}\cdot dt + \int_0^1 Q(x(t),y(t))\frac{dy}{dt}\,dt,
\end{aligned}$$

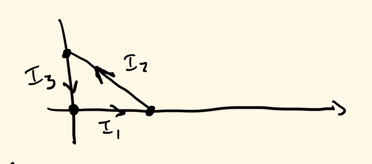
where $x(t) = e^t\cos(t)$, $y(t) = e^t\sin(t)$, and
$$\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = r'(t) = \left\langle e^t(\cos(t) - \sin(t)), e^t(\cos(t)+\sin(t)) \right\rangle,$$
and
$$x^2 + y^2 = e^{2t}(\cos^2 + \sin^2) = e^{2t}.$$
So, $(x^2+y^2)^{3/2} = e^{3t}$.

Thus,
$$\begin{aligned}
\int_C F\cdot dr &= \int_0^1 \frac{\cancel{e^t}\cos(t)}{e^{3t}}\left(\cancel{e^t}(\cos(t)-\sin(t))\right)dt + \int_0^1 \frac{e^t\sin(t)}{e^{3t}}\left(e^t(\cos(t)+\sin(t))\right) \\
&= \int_0^1 e^{-t}\Big[\cos^2(t) \underbrace{- \cancel{\cos(t)\sin(t)} + \cancel{\cos(t)\sin(t)}) + \sin^2(t)}_{=1}\Big]dt \\
&= \int_0^1 e^{-t}\,dt = -e^{-t}\Big|_0^1 = -e^{-1} + e^0 \\
&= 1 - \frac{1}{e} \\
&= \boxed{\frac{e-1}{e}}
\end{aligned}$$

Remark: You would (of course) get the same answer, regardless of what form of $\int_C F\cdot dr$ used!

Ex. Evaluate $\int_C y^3dx - x^3dy$, where C is the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ oriented counterclockwise!

Picture:



$\leadsto$ Use that $\int_C = \int_{I_1} + \int_{I_2} + \int_{I_3}$.

We must then parametrize each curve.

I_1: $y = 0$, $0 \le x \le 1$. So, $I_1 = \langle t, 0\rangle$, $0 \le t \le 1$.
I_2: $y = 1 - x$, $0 \le x \le 1$. ~~So, $I_2 = \langle t, 1-t\rangle$, $0 \le t \le 1$.~~

But this is wrong!!! When $t = 0$, @ $(0,1)$; $t = 1$, @ $(1,0)$!

So it is incorrectly oriented!

Switching x and y will reverse the order: that is, take $I_2 = \langle 1-t, t\rangle$ so that
$I_2(0) = \langle 1, 0\rangle$,
$I_2(1) = \langle 0, 1\rangle$, as required.

I_3: $x = 0$, $0 \le y \le 1$, but moving <u>down</u>. So, $I_3 = \langle 0, 1-t\rangle$, $0 \le t \le 1$, so that $I_3(0) = \langle 0, 1\rangle$,
$I_3(1) = \langle 0, 0\rangle$, as required.

So,
$$\begin{aligned}
\int_C y^3dx - x^3dy &= \int_{I_1+I_2+I_3} y^3dx - x^3dy \\
&= \int_{I_1} y^3dx - x^3dy + \int_{I_2} y^3dx - x^3dy + \int_{I_3} y^3dx - x^3dy \\
&= \int_0^1 0\cdot\frac{dx}{dt}\,dt - t^3\cdot\frac{dy}{dt}\,dt \\
&\quad + \int_0^1 t^3\frac{dx}{dt}dt - (1-t)^3\frac{dy}{dt}\,dt \\
&\quad + \int_0^1 (1-t)^3\frac{dx}{dt}\,dt - 0 \\
&= -\int_0^1 t^3\cdot 0 + \int_0^1 t^3(-1)\,dt - (1-t)^3\cdot 1\,dt + \int_0^1 (1-t)^3\cdot 0 \\
&= -\int_0^1 t^3\,dt - \int_0^1 (1-t)^3\,dt \\
&= -\frac{t^4}{4}\Big|_0^1 + \frac{(1-t)^4}{4}\Big|_0^1 \\
&= -\frac{1}{4} + 0 + 0 - \frac{1}{4} \\
&= \boxed{-\frac{1}{2}}
\end{aligned}$$

Question: Does there exist a scalar field F s.t. $\nabla F = \langle y^3, -x^3\rangle$?
If such a field F exists, then
$$\frac{\partial F}{\partial x} = y^3 \quad \text{and}$$
$$\frac{\partial F}{\partial y} = -x^3 . \quad \text{Solving these, we have}$$
$$\int \frac{\partial F}{\partial x}\,dx = \int y^3dx = xy^3 + f(y)$$
$$\int \frac{\partial F}{\partial y}\,dy = -\int x^3dy = -yx^3 + g(x).$$

So,
$$F(x,y) = xy^3 + f(y) = -yx^3 + g(x)$$
or
$$xy^3 + yx^3 = g(x) - f(y)$$
But, if $x = 0$, $f(y) = g(0) \equiv$ constant.
if $y = 0$, $g(x) = f(0) \equiv$ constant.

So, $F(x,y) = xy^3 + C = -yx^3 + C$
$$\Rightarrow xy^3 = -yx^3, \quad \rightarrow\leftarrow !$$

So there is no such vector field!