Topics:
- Parametric surfaces
- Surface integrals
- Vectors
- Stokes' & Divergence theorem

Parametric Surfaces, Surface Integrals & Flux

Similar to how we can parametrize a curve by a variable t, we can parametrize a surface by two variables u and v. Then the surface $r(u,v)$ is given by

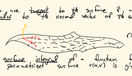
$$r(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle .$$

We then define the vectors

$$r_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

$$r_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

These vectors are tangent to the surface & with a perpendicular to the normal vector at the surface:



Then the surface integral of a function $f(x,y,z)$ over a smooth parametrized surface $r(u,v)$ is given by

$$\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v),y(u,v),z(u,v)) \cdot |r_u \times r_v| \, dA \qquad (*)$$

If the curve when $f=1$, we have the surface area of the surface $r(u,v)$.

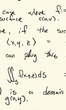
Furthermore, if the surface is given by $z = g(x,y)$, i.e. $\langle x,y,z \rangle = \langle x, y, g(x,y) \rangle$, then we can plug this into $(*)$ above to find

$$\iint_S f\,dS = \iint_D f(x,y,g(x,y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dA ,$$

where D is a domain in the (x,y) plane for the surface $g(x,y)$.

Ex. Find the surface area of the portion of the paraboloid $x^2 + y^2 = 6z$ inside the sphere $x^2+y^2+z^2 = 16 = 4^2$.

Picture:



So, we have $z = \frac{x^2+y^2}{6} = g(x,y)$, and D is given by $x^2+y^2 \le 4^2$.

Using the two eq^{n}'s, sub ① into ② to find

$6z + z^2 = 16 \iff z^2+6z-16 = 0$
$\iff (z+8)(z-2) = 0$
$\iff z = 2$

So, $\iint_S dS = \iint_D \sqrt{1+(g_x)^2+(g_y)^2}\, dA$, $\quad g_x = \frac{2x}{6} = \frac{x}{3}$, $\quad g_y = \frac{2y}{6} = \frac{y}{3}$;

$= \iint_D \sqrt{1+\left(\frac{x}{3}\right)^2+\left(\frac{y}{3}\right)^2}\, dA$.

Set $x = r\cos\theta$, $y = r\sin\theta$, $D = \{(r,\theta) : 0 \le \theta < 2\pi,\ 0 \le r^2 \le 16 - z^2 = 12\}$

Then: $\iint_S dS = \int_0^{2\pi}\int_0^{\sqrt{12}} \sqrt{1+\frac{r^2}{9}}\, r\,dr\,d\theta$; set $u = 1+\frac{r^2}{9}$, $du = \frac{2r}{9}dr \iff \frac{9}{2}du = r\,dr$

$= \frac{9}{2}\int_0^{2\pi}\int_1^{7/3} \sqrt{u}\, du\, d\theta$

$= \frac{9}{2}\cdot 2\pi \cdot \frac{2}{3}u^{3/2}\Big|_{t=1}^{t=7/3} = 6\pi\left(1+\frac{r^2}{9}\right)^{3/2}\Big|_{t=0}^{t=\sqrt{12}} = 6\pi\left(1+\frac{12}{9}\right)^{3/2} - 6\pi = \boxed{6\pi\left[\left(\frac{21}{9}\right)^{3/2} - 1\right]}$

Ex. Evaluate $\iint_S x\,dS$ where S is the upper hemisphere of radius a centered at the origin and $D(x,y) = x^2$.

Picture:

So, $x^2+y^2+z^2 = a^2 \Rightarrow z = \sqrt{a^2-(x^2+y^2)} =: g(x,y)$,

$g_x = \frac{-x}{z}$, $\quad g_y = \frac{-y}{z}$

Then, $\iint_S D\,dS = \iint_D x^2\sqrt{1+\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2}\, dA = \iint_D x^2\sqrt{\frac{x^2+y^2+z^2}{z^2}}\, dA = a\iint_D \frac{x^2}{z}\, dA$.

Remember: $D = \{(x,y) : x^2+y^2 \le a^2\}$. Polar coords! $\Rightarrow D = \{(r,\theta) : 0 \le r \le a,\ 0 \le \theta < 2\pi\}$.

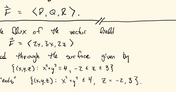
$z = \sqrt{a^2-r^2}$, and so

$\iint_S D\,dS = a\iint_D \frac{r^2\cos^2\theta}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta = a\int_0^{2\pi}\int_0^a \frac{r^3\cos^2\theta}{\sqrt{a^2-r^2}}\, dr\,d\theta$, set $u = a^2-r^2 \Rightarrow r^2 = a^2-u$, $-\frac{1}{2}du = r\,dr$

$= a\left(\int_0^{2\pi}\cos^2\theta\right)\cdot\left(\int_{a^2}^0 -\frac{1}{2}\left(\frac{a^2-u}{\sqrt{u}}\right)du\right)$

$= a\cdot\pi\cdot\frac{2a^3}{3} = \boxed{\frac{2\pi a^4}{3}}$

In addition to the surface integral of a function, we can talk about the surface integral of a vector field over a surface. That is, a vector field can be defined over the surface S, and talk about the "amount" (given by F) of $\vec{F}$ flux through the surface S.



The flux of F through S is given by

$$\iint_S F \cdot dS = \iint_S (F \cdot \hat{n})\, dS ,$$

where $\hat{n}$ is the unit normal vector to the surface S. That is, $\hat{n} = \frac{r_u \times r_v}{|r_u \times r_v|}$.

If $z = g(x,y)$, then $\hat{n} = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, 1 \right\rangle$, and so

$$\iint_S F \cdot dS = \iint_D \vec{F} \cdot \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1 \right\rangle dA = \iint_D \left[-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right] dA ,$$

where $\vec{F} = \langle P, Q, R \rangle$.

Ex. Find the flux of the vector field $\vec{F} = \langle 2z, 3x, 2y \rangle$ outward through the surface given by $\{(x,y,z) : x^2+y^2 = 4,\ -2 \le z \le 3\}$ and the "ends" $\{(x,y,z) : x^2+y^2 \le 4,\ z = -2, 3\}$.

Picture:

Break into 4 pieces: top & bottom (i.e. $x^2+y^2 \le 4$, $z = -2$ or $z = 3$), two sides (i.e. $0 \le \theta \le \pi$ and $\pi \le \theta < 2\pi$).

① Top: $z = 3$, $x^2+y^2 \le 4$, $\hat{n} = \langle 0,0,1 \rangle$ (outward, so oriented up!)

So, $\iint_{S_{top}} F \cdot dS = \iint_{S_{top}} \langle 2z, 3x, 2y \rangle \cdot \langle 0,0,1 \rangle \cdot \sqrt{1+0^2+0^2}\, dx\,dy = \iint_{x^2+y^2\le 4} 2z\, dx\,dy = \int_0^{2\pi}\int_0^2 2\cdot 3\, r\,dr\,d\theta = 6\cdot 2\pi \cdot \frac{r^2}{2}\Big|_0^2 = \boxed{24\pi}$

② Bottom: $z = -2$, $x^2+y^2 \le 4$, $\hat{n} = \langle 0,0,-1 \rangle$ (outward, so down!)

So, $\iint_{S_{bottom}} F \cdot dS = \iint_{S_{bottom}} -2z\, dx\,dy = 4\int_0^{\pi/2}\int_0^2 r\,dr\,d\theta = 4\cdot 2\pi \cdot \frac{r^2}{2}\Big|_0^2 = \boxed{16\pi}$

③ S_+ : $x^2+y^2 = 4 \Rightarrow y = \pm\sqrt{4-x^2}$ $\rightsquigarrow$ choose +ve, $-2 \le z \le 3$, $-2 \le x \le 2$

Then, $\hat{n} = \frac{\nabla g}{|\nabla g|} = \frac{\langle \frac{2x}{2y}, \frac{2y}{2y}, 0 \rangle}{\sqrt{\left(\frac{x}{y}\right)^2+1^2}} = \frac{\langle x, y, 0 \rangle}{2}$

So, $\iint_S F \cdot dS = \frac{1}{2}\iint_{S_+} \langle z, 3x, 2y \rangle \cdot \langle x, y, 0 \rangle \sqrt{1+0^2+\left(\frac{-x}{y}\right)^2}\, dx\,dz$

$= \frac{1}{2}\int_{-2}^3\int_{-2}^2 \left(z x + 3xy\right)\frac{2}{\sqrt{4-x^2}}\, dx\,dz = \int_{-2}^3\int_{-2}^2 \left(\frac{zx}{\sqrt{4-x^2}} + 3x\right) dx\,dz \rightsquigarrow$ trig subs. $x = 2\sin\theta \rightsquigarrow = \boxed{20\pi}$.

④ Repeat the process for $y = -\sqrt{4-x^2}$. Then,

$\iint_{S_-} F \cdot dS = \boxed{20\pi}$

Hence, the total flux is: $\iint_S F \cdot dS = 24\pi + 16\pi + 20\pi + 20\pi = \boxed{80\pi}$

Ex. Find the flux of the vector field $\vec{F} = \langle x, y, z \rangle$ upward through the hemisphere $z = \sqrt{a^2-x^2-y^2}$.

So, S is given by $\Phi = G(x,y,z) = x^2+y^2+z^2 = a^2$.

So, $\hat{n} = \frac{\langle \Phi_x, \Phi_y, \Phi_z \rangle}{|\langle \Phi_x, \Phi_y, \Phi_z \rangle|} = \frac{\langle 2x, 2y, 2z \rangle}{\sqrt{(2x)^2+(2y)^2+(2z)^2}} = \frac{\langle x, y, z \rangle}{a}$

Then, $F \cdot \hat{n} = \langle x, y, z \rangle \cdot \frac{\langle x, y, z \rangle}{a} = \frac{x^2+y^2+z^2}{a} = \frac{a^2}{a} = a$

$dS = \sqrt{1+\left(\frac{-x}{z}\right)^2+\left(\frac{-y}{z}\right)^2}\, dx\,dy = \frac{\sqrt{x^2+y^2+z^2}}{z}\, dx\,dy = \frac{a}{z}\, dx\,dy$

So, $\iint_S F \cdot dS = \iint_D a\,\frac{a}{z}\, dx\,dy = \iint_D \frac{a^2}{\sqrt{a^2-x^2-y^2}}\, dx\,dy$. $\Rightarrow$ Polar Coords, $0 \le \theta < 2\pi$, $0 \le r < a$.

$= a^2\int_0^{2\pi}\int_0^a \frac{1}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta$, sub $u = a^2-r^2$, $-\frac{1}{2}du = r\,dr$ $\rightsquigarrow$ $= \boxed{2\pi a^3}$

Stokes' Theorem: If C is a simple piecewise smooth curve bounding a surface S, then

$$\oint_C F \cdot dc = \iint_S (\operatorname{curl} F) \cdot n\, dS .$$

That is, Stokes' theorem relates a line integral over a curve C to a surface integral over a surface S.

Ex. Use Stokes' theorem to evaluate $\oint_C y\,dx - x\,dy + 2\,dz$, where C is the curve given by the intersection of the paraboloid $z = 4-x^2-y^2$ and cylinder $x^2+y^2 = 1$. $\longrightarrow$ $z = 4-(x^2+y^2) = 4-1 = 3$.

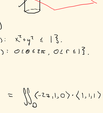
First, $\hat{n} = \frac{\langle 2x, 2y, -1 \rangle}{|\langle 2x, 2y, -1 \rangle|} = \frac{\langle -2x, -2y, -1 \rangle}{\sqrt{1^2+2^2(x^2+y^2)}} \rightsquigarrow \frac{\langle 2x, 2y, 1 \rangle}{\sqrt{1+4(x^2+y^2)}}$

and $dS = \sqrt{1+\left(-2x\right)^2+\left(-2y\right)^2}\, dx\,dy = \sqrt{1+4(x^2+y^2)}\, dx\,dy$

So, $\oint_C F \cdot dc = \iint_S (\nabla \times F) \cdot \hat{n}\, dS = \iint_S \langle 1,0,2 \rangle \cdot \frac{\langle 2x, 2y, 1 \rangle}{\sqrt{1+4(x^2+y^2)}} \sqrt{1+4(x^2+y^2)}\, dx\,dy = \int_0^{2\pi}\int_0^1 (2r\cos\theta - 2)\, r\,dr\,d\theta = \boxed{-2\pi}$

Ex. Use Stokes' theorem to evaluate $\oint_C 6x\,dx + 2x^2y\,dy + z^2\,dz$, where C is the curve given by the intersection of the plane $x+y+z = 6$ and the cylinder $x^2+y^2 = 1$, so that C runs counterclockwise when viewed from above.

$F = \langle z, x^2, z^2 \rangle$, picture:



and $\nabla \times F = \langle -2x, 1, 0 \rangle$

Then, C is given by $z = 6-x-y =: g(x,y)$

lying above the domain $D = \{(x,y) : x^2+y^2 \le 1\}$.

In polar coords, $D = \{(r,\theta) : 0 \le \theta < 2\pi,\ 0 \le r \le 1\}$.

Finally, $\hat{n} = \langle -g_x, -g_y, 1 \rangle = \langle 1, 1, 1 \rangle$.

Hence, $\oint_C F \cdot dc = \iint_D (\nabla \times F) \cdot \hat{n}\, dS = \iint_D \langle -2x, 1, 0 \rangle \cdot \langle 1, 1, 1 \rangle\, dS = \iint_D (1-2x)\, dx\,dy = \iint_D (1-2(6-x-y))\, dx\,dy = \iint_D (2x+2y-11)\, dx\,dy = \int_0^{2\pi}\int_0^1 \left(2(r\cos\theta + r\sin\theta) - 11\right) r\,dr\,d\theta = \boxed{-22\pi}$

Divergence Theorem: Let E be a bounded solid region in $\mathbb{R}^3$ with smooth boundary surface S. Then, if $\hat{n}$ is an outer unit normal for S and F is smooth,

$$\iint_S F \cdot dS = \iiint_E \operatorname{div} F\, dV .$$

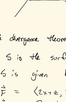
That is, the divergence theorem relates a surface integral to a volume integral.

Ex. Use the divergence theorem to compute $\iint_S \langle y, -x, z \rangle \cdot dS$ where S is the surface bounding the region above the cone $z = \sqrt{x^2+y^2}$ and below the plane $z = 3$.

So, $\iint_S F \cdot dS = \iiint_E \nabla \cdot F\, dV = \iiint_E \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle y, -x, z \rangle\, dV$

$= \iiint_E 1\, dV$. So, what is E?

Picture:



$\rightsquigarrow$ This is just the volume of a cone! So, $\int_0^{2\pi}\int_0^3\int_0^{z/3} r\,dr\,dz\,d\theta = \frac{1}{3}\pi\int_0^3 \frac{z^2}{9}\, dz = \frac{\pi}{27}\cdot\frac{z^3}{3}\Big|_0^3 = \boxed{9\pi}$

Ex. Use the divergence theorem to compute $\iint_S \langle 2x+z, 2xz+y, -y^2+3z \rangle \cdot dS$ where S is the surface of the sphere of radius 4 centered at $(-3,1,2)$.

So, S is given by $(x+3)^2+(y-1)^2+(z-2)^2 = 4^2$.

$\vec{F} = \langle 2x+z, 2xz+y, -y^2+3z \rangle$,

$\nabla \cdot F = 2+1+3 = 6$

Hence, $\iint_S F \cdot dS = \iiint_E \nabla \cdot F\, dV = 6\iiint_E dV$, sphere has volume $\frac{4\pi r^3}{3}$, and $r = 4$, so

$= 6 \cdot \frac{4\pi(4)^3}{3}$

$= 2 \cdot 2^6 \cdot 2^6\pi$

$= 2^7\pi$

$= \boxed{512\pi}$