Lab 10 Live

Thursday, June 4, 2020 8:02 PM

<u>Topic</u>: Fundamental Theorem for Line Integrals!

USRI's are out! Please fill out survey!

From last time: $F = \langle y^3, -x^3 \rangle$

Is there a function $f(x,y)$ such that

$$F = \nabla f.$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = y^3 \\ \dfrac{\partial f}{\partial y} = -x^3 \end{cases}$$

No!! (Not in this case!)

<u>Def^{n}</u>: If there is a function f so that

$$\nabla f = F,$$

for a given vector field F, then F is said to be <u>conservative</u>!

<u>FTfLI</u>: The fundamental theorem for line integrals says that if F is <u>conservative</u>, then

$$\int_C F \cdot dr = f(r(t)) \Big|_{t=\alpha}^{t=\beta},$$

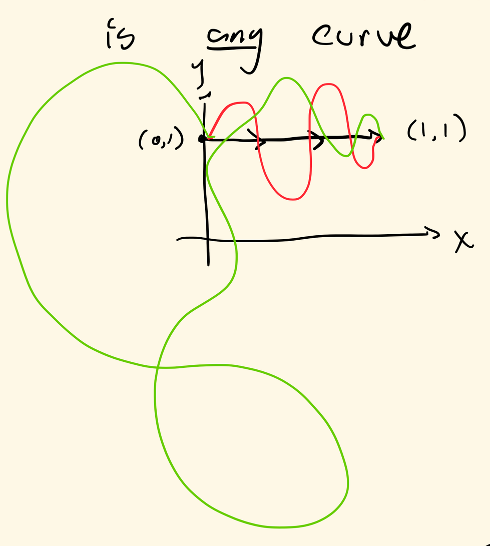
for any curve C parametrized by $r(t)$, $\alpha \le t \le \beta$.

Furthermore, if $r(t)$ is <u>closed</u> (i.e. $r(\alpha) = r(\beta)$), then

$$\int_C F \cdot dr = 0.$$

Ex.

Evaluate $\displaystyle\int_C \underbrace{\arctan(y)}_{P(x,y)}dx + \overbrace{\frac{x}{1+y^2}}^{Q(x,y)}dy$ where C is <u>any</u> curve from $(0,1)$ to $(1,1)$.



So, from $\int_C$,

$$F(x,y) = \langle P, Q \rangle = \left\langle \arctan(y), \frac{x}{1+y^2} \right\rangle.$$

Is this conservative? If so, $F = \nabla f$.

$$\begin{cases} \dfrac{\partial f}{\partial x} = \arctan(y) \\ \dfrac{\partial f}{\partial y} = \dfrac{x}{1+y^2} \end{cases}$$

$$\int^x \frac{\partial f}{\partial x}dx = \int^x \arctan(y)dx = x\arctan(y) + g(y).$$

$$\int^y \frac{\partial f}{\partial y}dy = \int^y \frac{x}{1+y^2}dy = x\arctan(y) + h(x)$$

So, $x\arctan(y) + g(y) = f(x,y) = x\arctan(y) + h(x)$

$\Rightarrow g(y) = h(x)$

$\Rightarrow g = h \equiv$ constant!

Then, take $C = 0$ so that

$$f(x,y) = x\arctan(y).$$

So, by the FTfLI,

$$\int_C F \cdot dr = f(r(t)) \Big|_{r(t)=(0,1)}^{r(t)=(1,1)}$$

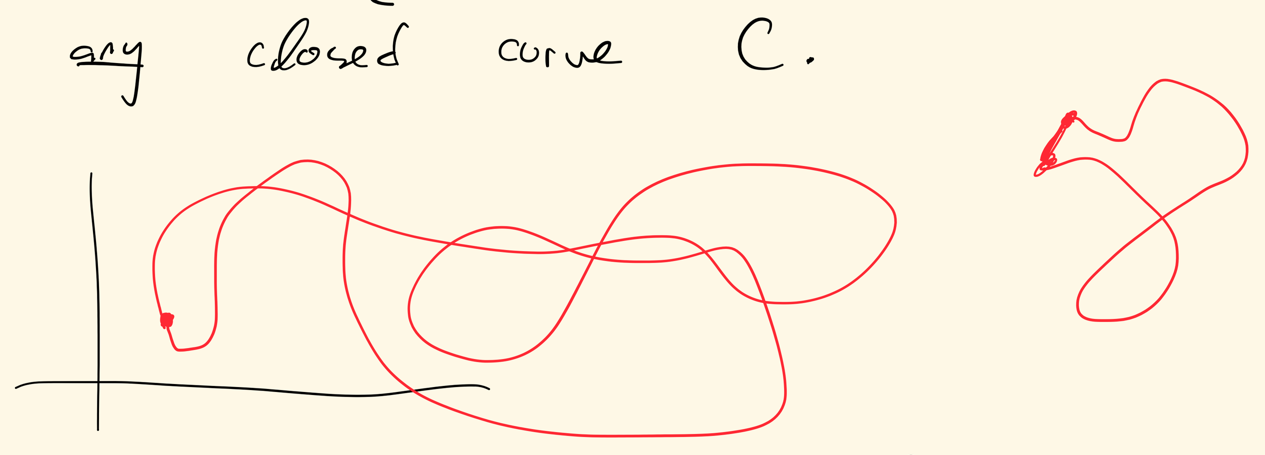
$$= x\arctan(y) \Big|_{(x,y)=(0,1)}^{(x,y)=(1,1)}$$

$$= 1 \cdot \arctan(1) - 0 \cdot \arctan(1)$$

$$= \boxed{\arctan(1)}$$

$$= \frac{\pi}{4}.$$

Ex. Show that $\displaystyle\int_C (4x^3 + 9x^2y^2)dx + (6x^3y + 6y^5)dy = 0$ for <u>any</u> closed curve C.



So, we seek $f(x,y)$ so that $\nabla f = F$, where

$$F(x,y) = \langle 4x^3 + 9x^2y^2, 6x^3y + 6y^5 \rangle$$

So,

$$\begin{cases} \dfrac{\partial f}{\partial x} = 4x^3 + 9x^2y^2 \\ \dfrac{\partial f}{\partial y} = 6x^3y + 6y^5 \end{cases}$$

$$f(x,y) = \int \frac{\partial f}{\partial x}dx = \int (4x^3 + 9x^2y^2)dx = x^4 + 3x^3y^2 + h(y)$$

$$f(x,y) = \int \frac{\partial f}{\partial y}dy = \int (6x^3y + 6y^5)dy = 3x^3y^2 + y^6 + g(x)$$

$\Rightarrow x^4 + \cancel{3x^3y^2} + h(y) = \cancel{3x^3y^2} + y^6 + g(x)$

$\Rightarrow x^4 - g(x) = y^6 - h(y).$

So, set $g(x) = x^4$, $h(y) = y^6$, all holds true!

i.e. $0 = 0$.

So, $f(x,y) = 3x^3y^2 + y^6 + g(x)$

$$= \boxed{3x^3y^2 + y^6 + x^4}$$

Hence, there exists $f(x,y)$ so that $F(x,y) = \nabla f$, and so F is a <u>conservative</u> vector field, and by the fundamental theorem for line integrals,

$$\int_C F \cdot dr = \int_C P\,dx + Q\,dy$$

$$= \int_C (4x^3 + 9x^2y^2)dx + (6x^3y + 6y^5)dy = 0,$$

since C is a closed curve.