Lab 10

Wednesday, June 3, 2020   12:57 PM

Topic: - Fundamental theorem for line integrals!

Def$^{=}$: If there is a function f so that $F = \nabla f$ for some vector field F, then F is called a conservative vector field.

The fundamental theorem for line integrals says that if F is a conservative vector field, then

$$\int_C F \cdot dr = f(r(t))\Big|_{\alpha}^{\beta} = f(r(\beta)) - f(r(\alpha))$$

where C is defined by $r(t)$ for $\alpha \le t \le \beta$.
This is to say, conservative vector fields are path independent! It doesn't matter which route you take, only the start and end point matter!
Furthermore, if the path is closed (i.e. $\alpha = \beta$), then $\int_C F \cdot dr = 0$!

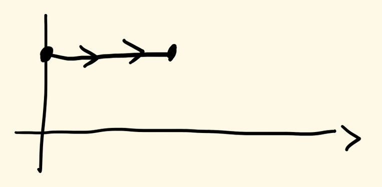
Ex. From last time, we asked if $\exists f$ so that $\nabla f = (y^3, -x^3)$.

This is impossible, since our curve C was closed, but $\int y^3 dx - x^3 dy \neq 0$.

One can show this directly as well.

Ex. Evaluate $\int_C \arctan(y) dx + \frac{x}{1+y^2} dy$ where C is any curve from $(0,1)$ to $(1,1)$.

Picture:



Since this must be solved for any curve, we should look to see if

$$F(x,y) = \langle P(x,y), Q(x,y) \rangle$$
$$= \langle \arctan(y), \frac{x}{1+y^2} \rangle$$

is conservative so we can apply the FTfLI!

So, we seek $f(x,y)$ so that $F = \nabla f$.

i.e.

$$\begin{cases} \frac{\partial f}{\partial x} = \arctan(y) \\ \frac{\partial f}{\partial y} = \frac{x}{1+y^2} \end{cases}$$

but, from the first eq^{n},

$$f(x,y) = x\arctan(y) + g(y).$$

From the second eq^{n},

$$f(x,y) = x\arctan(y) + h(x).$$

Hence, $g(y) = h(x) \Rightarrow g \equiv h \equiv$ constant, and so

$$f(x,y) = x\arctan(y) + C.$$

Let's choose $C = 0$ for simplicity.

Then, $r(t)$ must have $r(\alpha) = \langle 0, 1 \rangle$, $r(\beta) = \langle 1, 1 \rangle$.

Thus, by the FTfLI,

$$\int_C F \cdot dr = f(r(t))\Big|_{t=\alpha}^{t=\beta}$$
$$= f(r(\beta)) - f(r(\alpha))$$
$$= x\arctan(y)\Big|_{(x,y)=(1,1)} - x\arctan(y)\Big|_{(x,y)=(0,1)}$$
$$= 1 \cdot \arctan(1) - 0$$
$$= \boxed{\arctan(1)}$$

Ex. Show that $\int_C (4x^3 + 9x^2y^2) dx + (6x^3y + 6y^5) dy = 0$ for any closed curve C.

Since it is any closed curve, seek $f(x,y)$ so that $\nabla f = F = \langle 4x^3 + 9x^2y^2, 6x^3y + 6y^5 \rangle$

So,

$$\begin{cases} \frac{\partial f}{\partial x} = 4x^3 + 9x^2y^2 \\ \frac{\partial f}{\partial y} = 6x^3y + 6y^5 \end{cases}$$

Solving these,

$$\begin{cases} f(x,y) = x^4 + 3x^3y^2 + g(y) \\ \text{and} \\ f(x,y) = 3x^3y^2 + y^6 + h(x) \end{cases}$$

So,

$$x^4 + \cancel{3x^3y^2} + g(y) = \cancel{3x^3y^2} + y^6 + h(x)$$
$$x^4 + g(y) = y^6 + h(x).$$

So, set $h(x) = x^4$, $g(y) = y^6$, and hence

$$f(x,y) = x^4 + y^6 + 3x^3y^2.$$

Thus, by the FTfLI, F is a conservative vector field over a closed curve C, and so

$$\int_C F \cdot dr = 0 \quad !$$