Lab 8

Wednesday, May 27, 2020 2:55 PM

Topics: - triple integrals!
→ cartesian coords
→ introduce cylindrical coords

Ex. $\int_0^1\int_x^{2x}\int_0^y 2xyz\,dz\,dy\,dx$. → no setup required.

$= 2xy\int_0^y z\,dz = \cancel{2}xy\cdot\left(\frac{1}{\cancel{2}}z^2\Big|_{z=0}^{z=y}\right) = xy^3$

$= \int_0^1\int_x^{2x} xy^3\,dy\,dx$

$= x\int_x^{2x} y^3\,dy = x\left(\frac{y^4}{4}\Big|_{y=x}^{y=2x}\right) = \frac{x}{4}\left((2x)^4 - x^4\right) = \frac{15x^5}{4}$

$= \int_0^1 \frac{15x^5}{4}\,dx = \frac{15}{4}\cdot\frac{1}{6}x^6\Big|_{x=0}^{x=1} = \boxed{\frac{5}{8}}$

Ex. $\iiint_E 2x\,dV$ where $E = \{(x,y,z) : 0 \le x \le \sqrt{4-y^2},\ 0 \le y \le 2,\ 0 \le z \le y\}$.

$\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^y 2x\,dz\,dx\,dy = \int_0^2\int_0^{\sqrt{4-y^2}} 2x\left(z\Big|_{z=0}^{z=y}\right)dx\,dy$

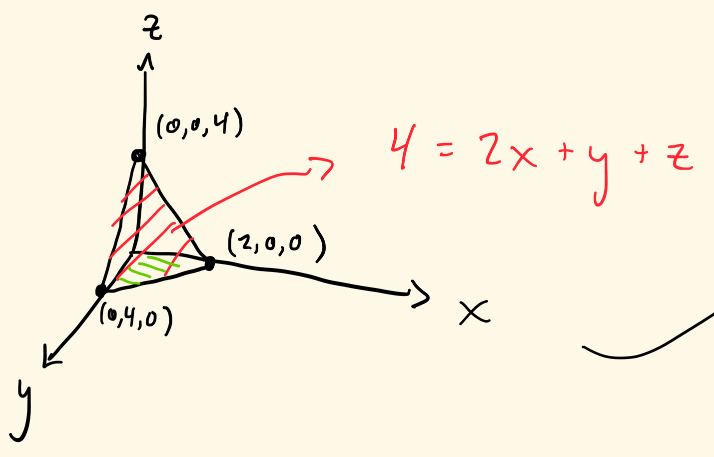
$= \int_0^2\int_0^{\sqrt{4-y^2}} 2xy\,dx\,dy = \int_0^2 y\left(x^2\Big|_{x=0}^{x=\sqrt{4-y^2}}\right)dy$

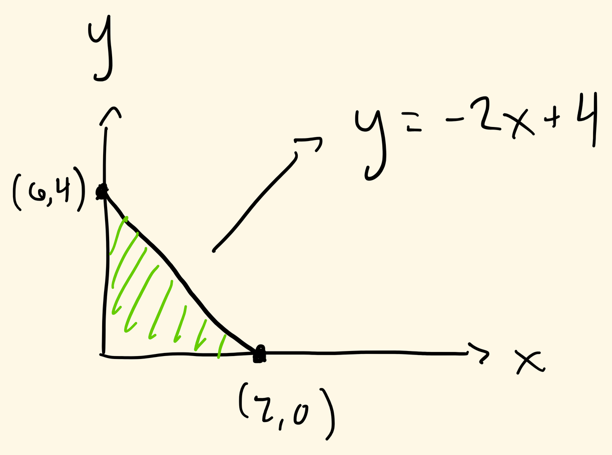
$= \int_0^2 y(4-y^2)\,dy = \left(2y^2 - \frac{y^4}{4}\right)\Big|_{y=0}^{y=2} = 2^3 - \frac{2^4}{2^2} = \boxed{4}$

Ex. Find the volume of the tetrahedron enclosed by the coordinate planes and the plane $2x+y+z = 4$.

So, $z = 4-2x-y$,

when $(x,y) = 0$, $z = 4$
when $(x,z) = 0$, $y = 4$
when $(y,z) = 0$, $x = 2$





So, $R = \{(x,y,z) : 0 \le x \le 2,\ 0 \le y \le 4-2x,\ 0 \le z \le 4-2x-y\}$

So, $\text{Volume} = \iiint_R dV = \int_0^2\int_0^{4-2x}\int_0^{4-2x-y} dz\,dy\,dx$

$= \int_0^2\int_0^{-2x+4}(4-2x-y)\,dy\,dx$

$= \int_0^2\left[(4-2x)y - \frac{y^2}{2}\right]\Big|_{y=0}^{y=-2x+4} dx$

$= \int_0^2\left[(4-2x)(4-2x) - \frac{(-2x+4)^2}{2}\right]dx$

$= \frac{1}{2}\int_0^2 (4-2x)^2\,dx$, set $u = 4-2x$, $-\frac{1}{2}du = dx$

$= -\frac{1}{4}\int_0^2 u^2\,dx$

$= -\frac{1}{4}\cdot\frac{u^3}{3}\Big|_{x=0}^{x=2} = -\frac{1}{12}\cdot(4-2x)^3\Big|_{x=0}^{x=2}$

$= 0 - \left(-\frac{1}{12}\right)\left(4^3\right)$

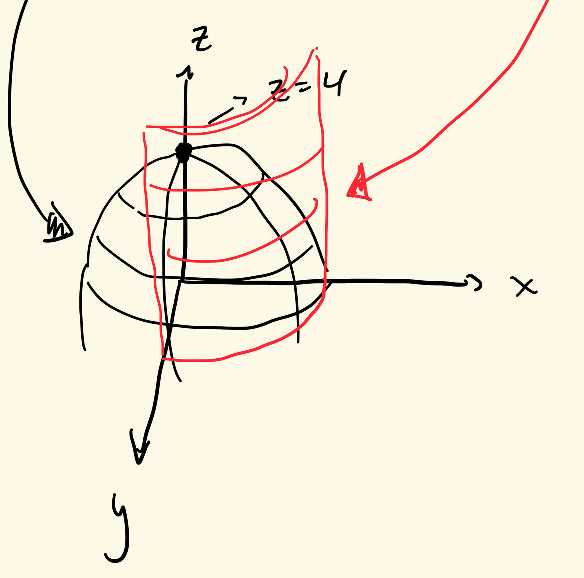
$= 16\cdot\frac{4}{12} = \boxed{\frac{16}{3}}$

Ex. Use cylindrical coords to find the volume of the region lying in the first octant, bounded above by

$z = 4-x^2-y^2$

and $x^2+y^2 = 2x$.

Picture:



, remember: $z = z$
$x = r\cos\theta$
$y = r\sin\theta$,
$x^2+y^2 = r^2$.

So, $0 \le z \le 4-x^2-y^2 = 4-r^2$
$0 \le r \le 2\cos\theta$
$0 \le \theta \le \pi/2$.

where does this come from? well, we know this cylinder is given by $x^2+y^2 = 2x$, and we want to be inside the cylinder, so look for (x,y) such that $x^2+y^2 \le 2x$. From the coordinate transform, $r^2 \le 2r\cos\theta$, or $r \le 2\cos\theta$.

So, $\iiint_E dV = \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}\int_0^{4-r^2} dz\,r\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}(4-r^2)\,r\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}}\left(2r^2 - \frac{r^4}{4}\right)\Big|_{r=0}^{r=2\cos\theta} d\theta$

$= \int_0^{\frac{\pi}{2}}\left(2\cos^2\theta - \frac{2^4}{4}\cos^4\theta\right)d\theta$

$= \int_0^{\frac{\pi}{2}}\left(2\cos^2\theta - 4\cos^4\theta\right)d\theta$.

Use the identity $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$,

⋮

$= \boxed{\frac{5\pi}{4}}$