Lab 7 Live

Topics - Double integrals!

Ex. $\int_0^2 \int_1^3 \left((2xy + x) \, dx \, dy \right)$

$= \int_0^2 \left(\int_1^3 (2xy + x) dx \right) dy$

$= \int_0^2 \left(\int_1^3 2xy \, dx + \int_1^3 x \, dx \right) dy$

$= \int_0^2 \left(y x^2 \Big|_{x=1}^{3} + \frac{x^2}{2} \Big|_{x=1}^{3} \right) dy$

$= \int_0^2 \left(y(3^2 - 1^2) + \left(\frac{3^2}{2} - \frac{1^2}{2} \right) \right) dy$

$= \int_0^2 \left(y(9-1) + \left(\frac{9}{2} - \frac{1}{2} \right) \right) dy$

$= \left[(9-1) \frac{y^2}{2} + \left(\frac{9-1}{2} \right) y \right]_{y=0}^{y=2}$

$= \left[\left(\frac{8}{2} \right) y^2 + \left(\frac{8}{2} \right) y \right]_{y=0}^{y=2}$

$= \frac{8}{2}(2^2 + 2) - 0$

$= 4 \cdot 6 - \frac{12}{2} = 24 - 6 = \boxed{18}$

Ex. $\int_R x^2 y \, dA$, where $R = [0,2] \times [0,3]$.

Ex. $\int_0^1 \left(\int_{x^2}^{x} xy \, dy \right) dx$.

Constant x^2 to x

$\int_0^x xy \, dy = x \int_0^x y \, dy = x \cdot \left(\frac{y^2}{2} \right) \Big|_{y=0}^{y=x} = \frac{x^3}{2}$.

$\int_0^1 \left(\frac{x^3}{2} \right) dx = \frac{x^4}{2 \cdot 4} \Big|_{x=0}^{x=1} = \boxed{\frac{1}{8}}$

Ex. $\int_0^2 \int_0^1 e^{y^2} \, dA$, where $D = \{(x,y) : 0 \le x \le y, \ 0 \le y \le 1\}$.

Try to integrate $\int_0^2 e^{y^2} ds$

Constant 0 to y

$\int_0^y e^{y^2} dx = e^{y^2} \int_0^y dx = e^{y^2} \left(x \right) \Big|_{x=0}^{x=y} = y e^{y^2}$.

$\int_0^1 y e^{y^2} dy$ $\quad u = y^2$, $\frac{1}{2} du = y \, dy$

$\frac{1}{2} \int_0^1 e^u du = \frac{1}{2} e^u \Big|_{y=0}^{y=1} = \frac{1}{2} e^{y^2} \Big|_{y=0}^{y=1} = \frac{1}{2} \left(e^1 - e^0 \right) = \boxed{\frac{e-1}{2}}$

Ex. $\int_0^2 \int_{x^3}^{8} \frac{1}{y^4 + 1} \, dy \, dx \neq \int_{y=0}^{2} \int_0^8 \frac{1}{y^4+1} \, dx \, dy$

Draw the region!

$0 \le x \le 8$

$x^{1/3} \le y \le 2$

$8^{1/3} = 2$.

First: Integrate in x from 0 to y^3.
Then: Integrate in y from 0 to 2.

Try $\int_0^2 \frac{1}{y^4+1} dy$

$\int_0^8 \int_{x^{1/3}}^{2} \frac{1}{y^4+1} dy \, dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1} dx \, dy$

$= \int_0^2 \left(\frac{1}{y^4+1} \left(\int_0^{y^3} dx \right) \right) dy$

$= \int_0^2 \left(\frac{1}{y^4+1} \right) \left(x \Big|_{x=0}^{x=y^3} \right) dy$

$= \int_0^2 \frac{1}{y^4+1} \cdot (y^3 - 0) \, dy = \int_0^2 \frac{y^3}{y^4+1} dy$

Set $u = y^4 + 1$, $\frac{1}{4} du = y^3 dy$

$\frac{1}{4} \int_0^2 \frac{du}{u} = \frac{1}{4} \ln(u) \Big|_{y=0}^{y=2} = \frac{1}{4} \ln(y^4+1) \Big|_{y=0}^{y=2} = \frac{1}{4} \ln(2^4+1) = \boxed{\frac{1}{4} \ln(17)}$

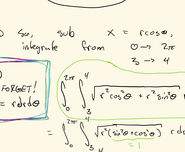
Polar Coords!

Ex. $\iint_R \sqrt{x^2+y^2} \, dA$, where R is the region between $x^2 + y^2 = 9 = 3^2$ and $x^2 + y^2 = 16 = 4^2$

$\iint_R x \, dA$

r: 3 to 4

θ: 0 to 2π



So, sub $x = r\cos\theta$, $y = r\sin\theta$

integrate from $0 \to 2\pi$ in θ

$3 \to 4$ in r.

AND DON'T FORGET! $dx\,dy = r\,dr\,d\theta$

$\int_0^{2\pi} \int_3^4 \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \, r \, dr \, d\theta$

$= \int_0^{2\pi} \int_3^4 \sqrt{r^2(\sin^2\theta + \cos^2\theta)} \, r \, dr \, d\theta$ ($\sin^2\theta + \cos^2\theta = 1$)

$= 2\pi \int_3^4 r^2 \, dr$

$= 2\pi \left(\frac{r^3}{3} \right) \Big|_3^4 = \frac{2\pi}{3} \left(4^3 - 3^3 \right)$

$= \boxed{\frac{74\pi}{3}}$

Ex. A thin plate lying in the xy plane is bounded by the curves $\begin{cases} \frac{x^2}{4} + \frac{y^2}{9} = 12 \\ x = 4y^2 \end{cases}$ $\longrightarrow$ $y = \pm\sqrt{12 - \frac{x^2}{4}} = \pm\frac{1}{2}\sqrt{12 - x^2}$, $y = \pm\frac{1}{4}\sqrt{x}$.

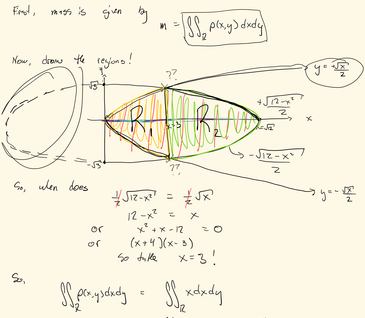
$x = \sqrt{12} \Rightarrow y = 0$

$x = 0, \ y = \sqrt{3}$

with mass density $\rho(x,y) = x$.
What is the mass of the plate?

Final, mass is given by $m = \iint_R \rho(x,y) \, dx \, dy$

Now, draw the regions!



So, when does $\frac{1}{2}\sqrt{12 - x^2} = \frac{1}{2}\sqrt{x}$

$12 - x^2 = x$

or $x^2 + x - 12 = 0$

or $(x+4)(x-3) = 0$

so double $x = 3$!

So,

$\iint_R \rho(x,y) \, dx \, dy = \iint_R x \, dx \, dy$

$= \iint_{R_1} x \, dx \, dy + \iint_{R_2} x \, dx \, dy$,

where $R_1 = \{(x,y) : -\frac{\sqrt{x}}{2} \le y \le \frac{\sqrt{x}}{2}, \ 0 \le x \le 3\}$

$R_2 = \{(x,y) : -\frac{\sqrt{12-x^2}}{2} \le y \le \frac{\sqrt{12-x^2}}{2}, \ 3 \le x \le \sqrt{12}\}$

$\int_0^3 \int_{-\frac{\sqrt{x}}{2}}^{\frac{\sqrt{x}}{2}} x \, dy \, dx + \int_3^{\sqrt{12}} \int_{-\frac{\sqrt{12-x^2}}{2}}^{\frac{\sqrt{12-x^2}}{2}} x \, dy \, dx$

$= \int_0^3 x \left(y \Big|_{y=-\frac{\sqrt{x}}{2}}^{y=\frac{\sqrt{x}}{2}} \right) dx + \int_3^{\sqrt{12}} x \left(y \Big|_{y=-\frac{\sqrt{12-x^2}}{2}}^{y=\frac{\sqrt{12-x^2}}{2}} \right) dx$

$= \int_0^3 x^{3/2} \, dx + \int_3^{\sqrt{12}} x\sqrt{12 - x^2} \, dx$

set $u = 12 - x^2$, $-\frac{1}{2} du = -x \, dx$

$\frac{2}{5} x^{5/2} \Big|_{x=0}^{x=3} + \int_0^{\sqrt{12}} \left(-\frac{1}{2} \right) \sqrt{u} \, du$

$= \frac{2}{5} 3^{5/2} - \frac{1}{2} u^{5/2} \cdot \frac{2}{5} \Big|_{x=3}^{x=\sqrt{12}}$

$= \frac{2}{5} 3^{5/2} - \frac{1}{5} (12 - x^2)^{5/2} \Big|_{x=3}^{x=\sqrt{12}}$

$= \frac{2}{5} 3^{5/2} + \frac{1}{5} (12 - 3)^{5/2}$

$= \boxed{\frac{23\sqrt{3}}{5}}$