Lab 7
Monday, May 25, 2020 3:58 PM

Topics: - double integrals
→ review stuff
→ changing the order of integration
→ polar coordinates
→ center of mass (application!)

Ex. $\int_0^3\int_{-2}^{5}(2xy+x)\,dx\,dy$

Remember, double integrals (or more!) should be read $\int_0^3\left(\int_{-2}^{5}(2xy+x)\,dx\right)dy$, so we'll hold and perform the first integration in the variable x:

$$\int_{-2}^{5}(2xy+x)\,dx = \left(x^2y+\frac{x^2}{2}\right)\Big|_{x=-2}^{x=5} = \left(5^2y+\frac{5^2}{2}-\left((-2)^2y+\frac{(-2)^2}{2}\right)\right) = \left((25-4)y+\frac{25}{2}-2\right) = 21y+\frac{21}{2}$$

Now we finish with the integration in the variable y:

$$\int_0^3\int_{-2}^{5}(2xy+x)\,dx\,dy = \int_0^3\left(21y+\frac{21}{2}\right)dy = \left(\frac{21}{2}y^2+\frac{21}{2}y\right)\Big|_{y=0}^{y=3} = \frac{21}{2}\left(3^2+3\right) = \boxed{126}$$

Ex. $\iint_R x^2y^2\,dA$, $R=[0,a]\times[0,b]$

First, $\iint_R x^2y^2\,dA = \int_0^b\left(\int_0^a x^2y^2\,dx\right)dy$ $\left(= \int_0^a\left(\int_0^b x^2y^2\,dy\right)dx\right)$

Now we integrate, first in x then in y:

$$\int_0^a x^2y^2\,dx = \frac{x^3y^2}{3}\Big|_{x=0}^{x=a} = \frac{a^3}{3}y^2$$

Now in y: $\frac{a^3}{3}\int_0^b y^2 = \frac{a^3}{3}\left(\frac{y^3}{3}\Big|_{y=0}^{y=b}\right) = \boxed{\frac{a^3b^3}{9}}$

Ex. $\int_0^1\int_0^x xy\,dy\,dx$

This time, the bounds of integration depend on x. But we can still integrate twice as in the previous examples.

First, $\int_0^x xy\,dy = x\int_0^x y\,dy = x\left(\frac{y^2}{2}\Big|_{y=0}^{y=x}\right) = x\cdot\frac{x^2}{2} = \frac{x^3}{2}$.

Then, $\int_0^1\int_0^x xy\,dy\,dx = \frac{1}{2}\int_0^1 x^3\,dx = \frac{1}{2}\cdot\frac{x^4}{4}\Big|_0^1 = \boxed{\frac{1}{8}}$

Ex. $\iint_D e^{y^3}\,dA$, $D=\{(x,y): 0\le x\le y, 0\le y\le 1\}$.

This time, if we write our integral with bounds, we have:

$\int_0^1\int_0^y e^{y^3}\,dx\,dy$. Notice that the inside integral is in terms of x, while the function depends only on y! Hence,

$$\int_0^y e^{y^3}\,dx = e^{y^3}\int_0^y dx = e^{y^3}\left(x\Big|_{x=0}^{x=y}\right) = ye^{y^3}.$$

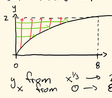
The second integral is now possible.

$$\int_0^1\int_0^y e^{y^3}\,dx\,dy = \int_0^1 ye^{y^3}\,dy, \quad \text{set } u=y^3,\ \frac{1}{3}du = y^2dy$$
$$= \frac{1}{3}\int_0^1 e^u\,du = \frac{1}{3}e^{y^3}\Big|_0^1 = \frac{1}{3}\left(e-1\right) = \boxed{\frac{e-1}{3}}$$

Ex. $\int_0^8\int_{y^{1/3}}^{2}\frac{1}{y^4+1}\,dy\,dx$. This time, the inside integral depends on y and the function also depends on y.

This is an example where changing the order of integration allows one to solve the integral! Try integrating $\int\frac{1}{y^4+1}\,dy$ ... it's difficult!

So, let's draw our bounds of integration: $\{(x,y): x^{1/3}\le y\le 2,\ 0\le x\le 8\}$.



Note: $8^{1/3}=2$!

First, integrate in y from $x^{1/3}\to 2$.
Second, integrate in x from $0\to 8$.

Alternatively, we can integrate in x from $0\to y^3$ (if $y=x^{1/3}$, $x=y^3$)
then, we can integrate in y from $0\to 2$.

That is,
$$D=\{(x,y): x^{1/3}\le y\le 2,\ 0\le x\le 8\}$$
$$=\{(x,y): 0\le x\le y^3,\ 0\le y\le 2\}.$$

So, $\int_0^8\int_{y^{1/3}}^{2}\frac{1}{y^4+1}\,dy\,dx = \int_0^2\int_0^{y^3}\frac{1}{y^4+1}\,dx\,dy$ (switched!!!)

Now we can integrate! $\int_0^{y^3}\frac{1}{y^4+1}\,dx = \frac{1}{y^4+1}\left(x\Big|_{x=0}^{x=y^3}\right) = \frac{y^3}{y^4+1}$.

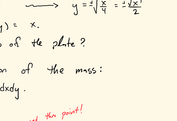
Now we can do the second integral! That is,

$\int_0^2\frac{y^3}{y^4+1}\,dx = \int_0^2\frac{y^3}{y^4+1}\,dy$, set $u=y^4+1$, $\frac{1}{4}du = y^3dy$.

Then, $\frac{1}{4}\int_1^{17}\frac{1}{u}\,du = \frac{1}{4}\ln(u)\Big|_1^{17} = \frac{1}{4}\ln(y^4+1)\Big|_0^2 = \frac{1}{4}\ln\left(2^4+1\right) = \boxed{\frac{1}{4}\ln(17)}$

Ex. $\iint_R\sqrt{x^2+y^2}\,dA$, R = the region between the curves $x^2+y^2=9=3^2$ and $x^2+y^2=16=4^2$

First, draw the region:



This is very challenging in terms of x & y. However, the region is perfect for polar coordinates! That is, the region above is easily described as $D=\{(r,\theta): 0\le\theta\le 2\pi,\ 3\le r\le 4\}$.

So, set $x=r\cos\theta$, $y=r\sin\theta$, and then

$\sqrt{x^2+y^2} = \sqrt{r^2(\cos^2\theta+\sin^2\theta)} = |r| = r$ (since $r>0$).

Also remember: $dx\,dy = r\,dr\,d\theta$! So,

Thus, $\iint_D\sqrt{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_3^4 r\cdot r\,dr\,d\theta$

$$= \int_0^{2\pi}\left(\frac{r^3}{3}\Big|_3^4\right)d\theta = \int_0^{2\pi}\left(\frac{4^3-3^3}{3}\right)d\theta = \frac{2\pi}{3}\left(4^3-3^3\right) = \frac{2\cdot 37\pi}{3} = \boxed{\frac{74\pi}{3}}$$

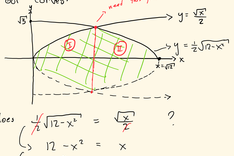
Ex. A thin plate lying in the xy-plane is bounded by the curves $x^2+4y^2=12 \longrightarrow y=\sqrt{\frac{12-x^2}{4}}=+\frac{1}{2}\sqrt{12-x^2}$ and $x=4y^2 \longrightarrow y=\pm\sqrt{\frac{x}{4}}=\pm\frac{\sqrt{x}}{2}$

with mass density $\rho(x,y)=x$.
What is the total mass of the plate?

First, we need the definition of the mass:

$$m = \iint_D\rho(x,y)\,dx\,dy.$$

Now, draw our curves:



So, when does $\frac{1}{2}\sqrt{12-x^2} = \frac{\sqrt{x}}{2}$?

$\to\ 12-x^2 = x$

$\to\ x^2+x-12=0$, so

$(x+4)(x-3)=0$,

and since we're looking for $x>0$, $x=3$ is our point!

Region ②: $\{(x,y): -\frac{\sqrt{x}}{2}\le y\le\frac{\sqrt{x}}{2},\ 0\le x\le 3\}$
Region ③: $\{(x,y): -\frac{1}{2}\sqrt{12-x^2}\le y\le\frac{1}{2}\sqrt{12-x^2},\ 3\le x\le\sqrt{12}\}$

Then,

$$\iint_D\rho(x,y)\,dx\,dy = \iint_{D_2}x\,dx\,dy + \iint_{D_3}x\,dx\,dy$$
$$= \int_0^3\int_{-\frac{\sqrt{x}}{2}}^{\frac{\sqrt{x}}{2}}x\,dy\,dx + \int_3^{\sqrt{12}}\int_{-\frac{\sqrt{12-x^2}}{2}}^{\frac{\sqrt{12-x^2}}{2}}x\,dy\,dx$$
$$= \int_0^3 x\cdot\left(y\Big|_{-\frac{\sqrt{x}}{2}}^{\frac{\sqrt{x}}{2}}\right)dx + \int_3^{\sqrt{12}}x\left(y\Big|_{-\frac{\sqrt{12-x^2}}{2}}^{\frac{\sqrt{12-x^2}}{2}}\right)dx$$
$$= \int_0^3 x\left(\sqrt{x}\right)dx + \int_3^{\sqrt{12}}x\sqrt{12-x^2}\,dx$$
$$= \frac{2}{5}x^{5/2}\Big|_0^3 + \left(-\frac{1}{2}\right)\int_3^{\sqrt{12}}\sqrt{u}\,du$$

(set $u = 12-x^2$, $du = -2x\,dx$, $x\,dx = -\frac{1}{2}du$)

$$= \frac{2}{5}\cdot 3^{5/2} + \left(-\frac{1}{2}\right)\left(\frac{2}{3}u^{3/2}\Big|_{x=3}^{x=\sqrt{12}}\right)$$
$$= \frac{2}{5}3^{5/2} + \left(-\frac{1}{\cancel{2}}\right)\left(\frac{\cancel{2}}{3}\right)\left(\left(12-x^2\right)^{3/2}\Big|_3^{\sqrt{12}}\right)$$
$$= \frac{2}{5}3^{5/2} + \left(-\frac{1}{3}\right)\left(0-\left(12-3^2\right)^{3/2}\right)$$
$$= \frac{2}{5}3^{5/2} + 3^{3/2}$$
$$= \sqrt{3}\left(\frac{2\cdot 3^2}{5}+1\right) = \boxed{\frac{2\cdot 3\sqrt{3}}{5}}.$$

ANOTHER POLAR EXAMPLE

Ex. Find the volume lying between the curves $z=x^2+y^2 \longrightarrow$ opens up! and $3z=4-x^2-y^2 \longrightarrow$ opens down!

First, sketch!

Using this, we can see the region (kind of...), and we can solve for the curve where these two surfaces meet. So,

take ③ − ⑫:
$$0 = 3(x^2+y^2)-4+x^2+y^2$$
$$= 4(x^2+y^2-1)$$

Hence, $x^2+y^2=1$, and since we are integrating over (x,y) inside this curve, polar coords seems like a good idea! That is, take $0\le\theta\le 2\pi$, and $0\le r\le 1$.

Then, $D=\{(r,\theta): 0\le\theta<2\pi,\ 0\le r\le 1\}$,
where $x=r\cos\theta$, $y=r\sin\theta$.

Now, write $f(x,y)=z$ using our two surfaces and convert to polar (don't forget that $dx\,dy = r\,dr\,d\theta$).

So, upper: $z=\frac{4}{3}-\frac{(x^2+y^2)}{3} = \frac{4}{3}-\frac{r^2}{3}$

lower: $z=x^2+y^2 = r^2$.

So,
$$\iint f(x,y)\,dx\,dy = \int_0^{2\pi}\int_0^1\left(\left(\frac{4}{3}-\frac{r^2}{3}\right)-r^2\right)r\,dr\,d\theta$$
$$= \frac{4}{3}\int_0^{2\pi}\int_0^1\left(r-r^3\right)dr\,d\theta$$
$$= \frac{4}{3}\cdot\left(2\pi\right)\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1$$
$$= \frac{8\pi}{3}\cdot\left(\frac{1}{2}-\frac{1}{4}\right)$$
$$= \boxed{\frac{2\pi}{3}}$$