Lab 6

Wednesday, May 20, 2020 12:07 PM

<u>Topics</u>: - Lagrange multipliers.

Sometimes we are given a function $f(x,y,z)$ which we wish to minimize (or maximize) given some additional restriction, or <u>constraint</u>.

The constraint is often expressed as $F(x,y,z) = 0$.

Then, the gradient of the function f must be <u>parallel</u> to the gradient of the constraint F. That is,

$$\nabla f = \lambda \nabla F,$$

for some real number λ. The constant λ is called the <u>Lagrange</u> <u>Multiplier</u> (anyone like zz top?)

If there are <u>two</u> constraints, we solve

$$\nabla f = \lambda \nabla F + \mu \nabla G,$$

where the constraints are $F(x,y,z) = 0 = G(x,y,z)$.

Ex. Say the temperature T at a point (x,y,z) is given by

$$T = 4xyz^2.$$

Find the <u>hottest</u> point on the sphere $x^2+y^2+z^2 = 100$.

<u>Sol</u>: hottest $\Rightarrow$ maximize, constraint $\Rightarrow$ Lagrange multipliers!

So, $f(x,y,z) = 4xyz^2$ (we want to maximize this)

and $F(x,y,z) = x^2+y^2+z^2-100 = 0$ (constraint in the form $F(x,y,z)=0$).

So,

$$\nabla f = (4yz^2, 4xz^2, 8xyz)$$
$$\nabla F = (2x, 2y, 2z)$$

Then,

$$\nabla f = \lambda \nabla F$$
$$\hookrightarrow (4yz^2, 4xz^2, 8xyz) = \lambda(2x, 2y, 2z)$$
$$\hookrightarrow \begin{aligned} 2\cancel{4}yz^2 &= \lambda \cancel{2}x \quad\rightarrow\quad 2yz^2 = \lambda x \quad (I)\\ 2\cancel{4}xz^2 &= \lambda\cancel{2}y \quad\rightarrow\quad 2xz^2 = \lambda y \quad (II)\\ 4\cancel{8}xyz &= \lambda\cancel{2}z \quad\rightarrow\quad 4xyz = \lambda z \quad (III)\end{aligned}$$

and also $x^2+y^2+z^2-100 = 0$ (IV)

So, we have <u>four</u> eq's and <u>four</u> unknowns! Now solve for all of these values!

If we solve for $2z^2$ in eq (I) & (II), we find

$$\lambda \frac{x}{y} = 2z^2 = \lambda \frac{y}{x}$$

and so $x^2 = y^2$ (assuming $x, y, z, \lambda \neq 0$!)

Then, from eq (III), $\lambda = 4xy$. Hence,

$$2z^2 = \lambda\frac{y}{x} = 4xy \cdot \frac{y}{x} = 4y^2.$$

If we sub these into our constraint, we have

$$\begin{aligned} 0 &= x^2+y^2+z^2-100\\ &= y^2+y^2+2y^2-100\\ \Rightarrow 100 &= 4y^2\\ \Rightarrow 25 &= y^2\\ \Rightarrow y &= \pm 5.\end{aligned}$$

Hence, $x = \pm 5$, and

$$z^2 = 2y^2 = 2\cdot 25 = 50$$
$$\Rightarrow z = \pm\sqrt{50} = \pm\sqrt{2}\cdot 5$$

Thus, $(x,y,z) = (\pm 5, \pm 5, \pm\sqrt{2}\cdot 5)$

So,

$$\begin{aligned} T &= 4(\pm 5)(\pm 5)\cdot(\pm\sqrt{2}\cdot 5)^2\\ &= 100\cdot 2\cdot 25\\ &= 5000.\end{aligned}$$

Of course, if $x=-5$, $y=-5$ is necessary (otherwise $T=-5000<0$).

So, our maximum temperature is 5000 at

$(x,y,z) = (5, 5, \pm\sqrt{2}\cdot 5)$ or

$(x,y,z) = (-5,-5, \pm\sqrt{2}\cdot 5)$

Ex. Show that the product of all angles of a triangle is largest for an equilateral triangle.

<u>Sol</u>: This problem requires a bit of setup. First, let (x,y,z) denote the angles of our triangle. Then, we maximize their <u>product</u>:

$$f(x,y,z) = xyz$$

Our constraint is given by the triangle itself. That is, we know $x+y+z = 180^\circ = \pi$ radians.

So, maximize $f(x,y,z) = xyz$

subject to $F(x,y,z) = x+y+z-\pi = 0$.

So,

$$\nabla f = (yz, xz, xy),$$
$$\nabla F = (1,1,1).$$

So,

$$\nabla f = \lambda\nabla F$$
$$\hookrightarrow (yz, xz, xy) = \lambda(1,1,1), \quad \text{and so}$$

$$\begin{cases} yz = \lambda\\ xz = \lambda\\ xy = \lambda\\ x+y+z-\pi = 0\end{cases}$$

From eq (I) & (II), $\frac{\lambda}{y} = z = \frac{\lambda}{x}$

$$\Rightarrow x = y.$$

Also, from (II) & (III) $\Rightarrow \frac{\lambda}{z} = x = \frac{\lambda}{y}$

$$\Rightarrow y = z.$$

Hence, $x = y = z$.

Then, $x+y+z-\pi = 3x-\pi = 0$

$$\Rightarrow x = \frac{\pi}{3}.$$

Thus, $xyz = \left(\frac{\pi}{3}\right)^3$ is the largest product of angles,

and since $x=y=z$, the triangle is equilateral!

Ex. A container is constructed in a cylindrical shape with top & bottom. If the surface area is <u>fixed</u>, find the height/radius which maximizes the volume.

So, first we see that we wish to maximize the volume:

$$V = \pi r^2\cdot h$$

subject to a <u>fixed</u> surface area:

$$SA = \underset{\text{top}}{\pi r^2} + \underset{\text{bottom}}{\pi r^2} + \underset{\text{centre}}{h\cdot 2\pi r}$$
$$= 2\pi r(r+h)$$

So,

$$f(r,h) = \pi r^2\cdot h$$
$$F(r,h) = 2\pi r(r+h) - S = 0,$$

where $S > 0$ is fixed.

Then,

$$\begin{aligned}\nabla f &= (2\pi r h, \pi r^2)\\ \nabla F &= (2\pi(r+h) + 2\pi r, 2\pi r)\\ &= (2\pi(2r+h), 2\pi r)\end{aligned}$$

Then,

$$\nabla f = \lambda\nabla F$$
$$\hookrightarrow (2\pi r h, \pi r^2) = \lambda(2\pi(2r+h), 2\pi r)$$

So,

$$\cancel{2\pi} r h = \lambda\cancel{2\pi}(2r+h)$$
$$\cancel{\pi} r^{\cancel{2}} = \lambda 2\cancel{\pi}\cancel{r}$$

or

$$\begin{cases} rh = \lambda(2r+h) & (I)\\ r = 2\lambda & (II)\\ 2\pi r(r+h) - S = 0 & (III)\end{cases}$$

and

From (II), $\lambda = r/2$. Sub into (I):

$$\cancel{r}h = \frac{\cancel{r}}{2}(2r+h)$$
$$\hookrightarrow 2h = 2r+h$$
$$\hookrightarrow h = 2r$$

Then, sub into eq (III):

$$\begin{aligned} S &= 2\pi r(r+h)\\ &= 2\pi r(r+2r)\\ &= 6\pi r^2\end{aligned}$$

So, $r = \pm\sqrt{\frac{S}{6\pi}}$

and

$$h = 2r = \pm 2\sqrt{\frac{S}{6\pi}}$$

Since this is a "real world" solution, r & h should be positive.

So, $(r,h) = \left(\sqrt{\frac{S}{6\pi}}, 2\sqrt{\frac{S}{6\pi}}\right)$, and the maximum volume is

$$\begin{aligned} V &= \pi r^2 h\\ &= \pi\left(\sqrt{\frac{S}{6\pi}}\right)^2\cdot 2\sqrt{\frac{S}{6\pi}}\\ &= \pi\left(\frac{S}{6\pi}\right)\cdot 2\sqrt{\frac{S}{6\pi}}\\ &= 2\pi\left(\frac{S}{6\pi}\right)^{3/2}.\end{aligned}$$