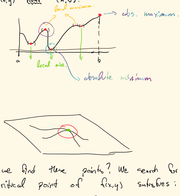
Topic: - maximum / minimum (in 2-D!)

Def: Given a function $z = f(x,y)$ and a point (a,b) in its domain,
- $f(a,b)$ is a **local maximum** (or **local minimum**) value if $f(a,b) \geq f(x,y)$ ($f(a,b) \leq f(x,y)$) when (x,y) is near (a,b).
- $f(a,b)$ is the **absolute maximum** (or **absolute minimum**) value if $f(a,b) \geq f(x,y)$ ($f(a,b) \leq f(x,y)$) when (x,y) is in the entire domain of f.

1-D:



How to we find these points? We search for **critical points**!
So, a critical point of $f(x,y)$ satisfies:

(i) $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$ (or DNE at (x_0,y_0))

(ii) Either f_x or f_y DNE at (x_0,y_0)

Notice: Every local max/min is a critical point, but NOT every critical point is a local max/min.

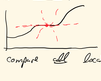
Second Derivative Test: Set $D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}^2(a,b)$

If:

(i) $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local **min**!

(ii) $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local **max**!

(iii) $D(a,b) < 0$, then (a,b) is neither max nor min, and (a,b) is a **saddle**!



So, to find the absolute max or min, compare all local max/mins and the boundaries!

Ex. Find & classify all crit. pts. of $f(x,y) = x^4 - 12xy + y^4$.

Proof, f is a poly^2 $\Rightarrow$ is continuous everywhere
$\Rightarrow$ no crit. pts. of type (ii).

Next, find derivatives:

$f_x(x,y) = 4x^3 - 12y$, $f_{xx} = 12x^2$

$f_{xy} = -12$ $(= f_{yx})$

$f_y = -12x + 4y^3$, $f_{yy} = 12y^2$

If $f_x = f_y = 0$,

$0 = 4x^3 - 12y$ ①

$0 = 4y^3 - 12x$ ②

Next, from ①:

$12y = 4x^3$

$y = \frac{4}{12}x^3 = \frac{1}{3}x^3$,

$y^3 = \left(\frac{1}{3}x^3\right)^3 = \frac{x^9}{27}$

Sub into eq^n ②:

$12x = 4y^3$

$= \frac{4}{27} \cdot x^9$

Rearrange, solve for x:

$0 = \frac{4}{27}x^9 - 12x$

$= x\left(\frac{4}{27}x^8 - 12\right)$

$= \frac{4x}{27}\left(x^8 - 12\cdot\frac{27}{4}\right)$

$0 = \frac{4x}{27}\left(x^8 - 3\cdot 27\right)$

$\Rightarrow$ $x = 0$, or $x = \sqrt[8]{3\cdot 27}$

$= (81)^{1/8}$

$= (3\cdot 9\cdot 3)^{1/8}$

$= (3\cdot 3\cdot 3\cdot 3)^{1/8}$

$= 3^{4/8} = \sqrt{3}$

Now find y!

$0 = 4x^3 - 12y$

$y = \frac{1}{3}x^3$,

when $x = 0$: $y = \frac{1}{3}0^3 = 0$ $\Rightarrow$ $(0,0)$

when $x = \sqrt{3}$, $y = \frac{1}{3}(\sqrt{3})^3 = \frac{1}{3}3^{3/2} = \sqrt{3}$. $\Rightarrow$ $(\sqrt{3},\sqrt{3})$

when $x = -\sqrt{3}$, $y = \frac{1}{3}(-\sqrt{3})^3 = -\sqrt{3}$ $\Rightarrow$ $(-\sqrt{3},-\sqrt{3})$

So, our critical points are: $(0,0)$, $(\sqrt{3},\sqrt{3})$, $(-\sqrt{3},-\sqrt{3})$

Next, classify! We need $D(0,0)$, $D(\sqrt{3},\sqrt{3})$, $D(-\sqrt{3},-\sqrt{3})$:

$f_x(x,y) = 4x^3 - 12y$, $f_{xx} = 12x^2$

$f_{xy} = -12$ $(= f_{yx})$

$f_y = -12x + 4y^3$, $f_{yy} = 12y^2$

(1) $D(0,0) = f_{xx}(0,0)\cdot f_{yy}(0,0) - f_{xy}^2(0,0)$

$= 0 - (-12)^2$

$= -144 < 0$.

$f_{xx}(0,0) = 0$

(2) $D(\sqrt{3},\sqrt{3}) = 12(\sqrt{3})^2 \cdot 12(\sqrt{3})^2 - (144)$

$= 12^2 \cdot \underbrace{(3)(3)}_{>1} - 144$

> 144

> 0,

$f_{xx}(\sqrt{3},\sqrt{3}) = 12(\sqrt{3})^2 > 0$

(3) $D(-\sqrt{3},-\sqrt{3}) = 12(-\sqrt{3})^2 12(\sqrt{3})^2 - 144$

> 0

$f_{xx}(\sqrt{3},-\sqrt{3}) > 0$.

So, make a table:

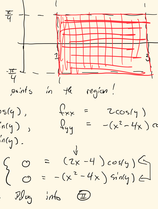
Critical Point	D	f_{xx}	Class
$(0,0)$	< 0	$= 0$	saddle!
$(\sqrt{3},\pm\sqrt{3})$	> 0	> 0	local min!

Ex. Find the absolute max/min of

$f(x,y) = (x^2 - 4x)\cos(y)$ on

$f(x,y)$: $1 \leq x \leq 3$, $-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$ }.

So,



First, find all critical pts in the region!

$f_x = (2x-4)\cos(y)$, $f_{xx} = 2\cos(y)$

$f_y = -(x^2-4x)\sin(y)$, $f_{yy} = -(x^2-4x)\cos(y)$

$f_{xy} = -(2x-4)\sin(y)$.

So, crit. pt. if:

$0 = (2x-4)\cos(y)$ ①

$0 = -(x^2-4x)\sin(y)$ ②

If $x = 2$, ① $= 0$. Plug into ②:

$\Rightarrow -(2^2-4\cdot 2)\sin(y) = 4\sin(y)$,

$= 0$ if $y = 0$!

So, $(2,0)$ is a critical point!

Similarly, if $x = 0$ or 4, ② $= 0$. But,

$-4\cos(y) \neq 0$, $4\cos(y) \neq 0$

So, $(2,0)$ is our only interior crit. pt.

Classify: $D(2,0) = 8 > 0$

$f_{xx}(2,0) = 2\cos(0) > 0$

$\Rightarrow (2,0)$ is a local min!

Now check boundaries! Set $x = 1$ $x = 3$

$y = \frac{\pi}{4}$ $y = \frac{\pi}{4}$,

and check as in 1-D!

(i) $f(1,y) = -3\cos(y)$.

$\frac{\partial f}{\partial y} = +3\sin(y) = 0$ when $y = 0$.

A crit. pt. of $f(1,y)$ is $y = 0$, or $(1,0)$.

(ii) $f(3,y) = -3\cos(y)$ $\rightsquigarrow$ $(3,0)$

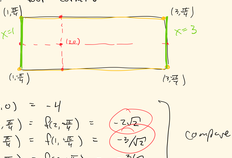
(iii) $f(x,\frac{\pi}{4})$ $\rightsquigarrow$ $(2,\frac{\pi}{4})$

(iv) $f(x,\frac{-\pi}{4})$ $\rightsquigarrow$ $(2,\frac{-\pi}{4})$

So, all crit. pts. are:

$(2,0)$ $(1,0)$ $(3,0)$ $(2,\frac{\pi}{4})$, $(2,\frac{-\pi}{4})$

+ four corners



So,

$f(2,0) = -4$

$f(2,\frac{\pi}{4}) = f(2,\frac{-\pi}{4}) = -2\sqrt{2}$

$f(1,\frac{\pi}{4}) = f(1,\frac{-\pi}{4}) = -3/\sqrt{2}$ } compare!

$f(3,\frac{\pi}{4}) = f(3,\frac{-\pi}{4}) = -3/\sqrt{2}$

$f(1,0) = -3$

$f(3,0) = -3$

Therefore, the ~~absolute~~ maximum is $-3/\sqrt{2}$

the absolute minimum is -4