Lab 4 Live

Thursday, May 14, 2020 8:02 PM

Topics: - the gradient
- directional derivatives.

Def^{n}: Given a function $f(x,y)$ with partial derivatives f_x, f_y. Then the gradient of f, ∇f, is

$$\underbrace{\nabla f := \langle f_x(x,y), f_y(x,y) \rangle}_{\text{A VECTOR !!}}$$

(If $f = f(x,y,z)$, then $\nabla f = \langle f_x, f_y, f_z \rangle$).

Ex. Find ∇f of $f(x,y) = xy + e^{x-y}$.

Sol^{n}: $f_x = y + e^{x-y}$

$f_y = x + e^{x-y} \cdot (-1) = x - e^{x-y}$.

So, $\nabla f = \langle y + e^{x-y}, x - e^{x-y} \rangle$.

Something to remember: The gradient always points in the direction of greatest increase!

Further, the magnitude is given by the mag. of ∇f,

$$|\nabla f| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

Def^{n}: The directional derivative of $f(x,y)$ in the direction $\vec{u} = (u_1, u_2)$ is

$$D_u f = u \cdot \nabla f,$$

where $|\vec{u}| = 1$.

Ex. Find the directional derivative of $f(x,y) = xy + e^{x-y}$ in the direction $\vec{u} = \langle 1, 2 \rangle$.

Sol^{n}: $\nabla f = \langle y + e^{x-y}, x - e^{x-y} \rangle$.

Then, $\vec{u} = \langle 1, 2 \rangle$, $|\vec{u}| = \sqrt{1^2 + 2^2} = \sqrt{5}$.

So,

$$D_u f = \frac{u}{|u|} \cdot \nabla f$$

$$= \frac{\langle 1, 2 \rangle \cdot \langle y + e^{x-y}, x - e^{x-y} \rangle}{\sqrt{5}}$$

$$= \frac{y + e^{x-y} + 2(x - e^{x-y})}{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}\left(2x + y - e^{x-y}\right).$$

Ex. Find the greatest rate of change of $f(x,y) = e^{x-y}$ at $(x,y) = (1,1)$. In which direction does this occur?

Remember! The gradient is related to the greatest R.o.C.!

$$\nabla f = \langle f_x, f_y \rangle$$

$$= \langle e^{x-y}, -e^{x-y} \rangle,$$ so the gradient at $(x,y) = (1,1)$ is

$$\nabla f(1,1) = \langle e^0, -e^0 \rangle$$

$$= \langle 1, -1 \rangle$$

Greatest R.o.C. @ $(1,1)$ is: $|\nabla f(1,1)| = \sqrt{1^2 + (-1)^2}$

$= \sqrt{2}$.

So, the greatest rate of change of $f(x,y) = e^{x-y}$ is $\sqrt{2}$ in the direction $\langle 1, -1 \rangle$.

Ex. $f(x,y) = \ln(x) y^2 - \sin(y)$. (i) Greatest R.o.C. @ $(x,y) = (e, \pi)$

(ii) Direction?

1. $\nabla f = \left\langle \frac{y^2}{x}, 2y \ln(x) - \cos(y) \right\rangle$

2. $\nabla f(x,y) = \nabla f(e, \pi)$

$= \left\langle \frac{\pi^2}{e}, 2\pi \ln(e) - \cos(\pi) \right\rangle$

$= \left\langle \frac{\pi^2}{e}, 2\pi + 1 \right\rangle$.

3. $|\nabla f| = \sqrt{\left(\frac{\pi^2}{e}\right)^2 + (2\pi + 1)^2}$

$= \sqrt{\frac{\pi^4}{e^2} + (2\pi + 1)^2}$

So, the Greatest R.o.C. of $f(x,y)$ at (e, π) is $\sqrt{\frac{\pi^4}{e^2} + (2\pi+1)^2}$ in the direction $\left\langle \frac{\pi^2}{e}, 2\pi + 1 \right\rangle$.

~~10~~ 5 mins left!

Don't forget time to upload!