Lab 4

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Topics: - Directional derivatives
- Gradients

Definition: Given a function $z = f(x,y)$, the gradient of $f(x,y)$, written $\nabla f$, is the vector

$$\boxed{\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle}.$$

Of course, we must assume these derivatives exist!

Ex. Find the gradient of $f(x,y) = xy + e^{x-y}$

First, $\frac{\partial f}{\partial x} = y + e^{x-y}$

and $\frac{\partial f}{\partial y} = x - e^{x-y}$

Hence,
$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$
$$= \left\langle y + e^{x-y}, x - e^{x-y} \right\rangle.$$

Something to remember: The maximum rate of change of the function $f(x,y)$ is given by the magnitude of the gradient,

i.e.
$$|\nabla f| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2},$$

since $\nabla f$ is a vector!

We can also talk about the rate of change in other directions as well.
Given a direction vector $\vec{u} := \langle u_1, u_2 \rangle$, the Directional Derivative of $f(x,y)$ in the direction $\vec{u}$ is given by $\boxed{D_u f = \vec{u} \cdot \nabla f}$

Ex. Find the directional derivative of $f$ in the direction $\vec{u} = \langle 1, 2 \rangle$, where $f(x,y) = xy + e^{x-y}$.
We already found
$$\nabla f = \left\langle y + e^{x-y}, x - e^{x-y} \right\rangle.$$

Then,
$$D_u f = \vec{u} \cdot \nabla f$$
$$= \langle 1, 2 \rangle \cdot \left\langle y + e^{x-y}, x - e^{x-y} \right\rangle$$
$$= y + e^{x-y} + 2(x - e^{x-y})$$
$$= 2x + y - e^{x-y}.$$

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Ex. Find the greatest rate of change of the function $f(x,y) = e^{x-y}$ at $(x,y) = (1,1)$. In which direction does this occur?

Remember: The gradient gives the greatest rate of change!

So, $\nabla f = \langle e^{x-y}, -e^{x-y} \rangle$, and so the gradient at $(1,1)$ is:

$$\nabla f(1,1) = \langle e^0, -e^0 \rangle = \langle 1, -1 \rangle.$$

The magnitude is then $\sqrt{1^2 + (-1)^2} = \sqrt{2}$.

The direction this occurs is then given by the normalized gradient
$$\frac{\nabla f}{|\nabla f|} = \frac{\langle 1, -1 \rangle}{\sqrt{2}}$$

(Note that the direction is still given by just $\nabla f$).

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Ex. Find the greatest rate of change of $f(x,y) = \ln(x) \cdot y^2 - \sin(y)$ at $(x,y) = (e, \pi)$

First, $\nabla f = \left\langle \frac{y^2}{x}, 2\ln(x) y - \cos(y) \right\rangle$

So,
$$\nabla f(e,\pi) = \left\langle \frac{\pi^2}{e}, 2\ln(e)\pi - \cos(\pi) \right\rangle$$
$$= \left\langle \frac{\pi^2}{e}, 2\pi + 1 \right\rangle$$

The magnitude is then $|\nabla f| = \sqrt{\frac{\pi^4}{e^2} + (2\pi + 1)^2}$

This occurs in the direction of $\nabla f$, i.e. $\left\langle \frac{\pi^2}{e}, 2\pi + 1 \right\rangle$.