Lab 3 Live

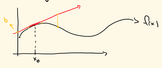
Tuesday, May 12, 2020 8:07 PM

Topics: - Tangent plane / linear approx.
- the 4 kinds of critical / important points.

Recall: $f(x,y)$, what is $\frac{\partial f}{\partial y}$?

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Tangent Plane: Recall in 1-D, we have the tangent line!



Near x_0, $f(x) \sim mx + b$

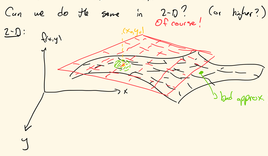
Our Taylor Expansion! $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

$f(x) \sim f(x_0) + f'(x_0)(x-x_0)$

Can we do the same in 2-D? (or higher?)

Of course!

2-D:



Equation for a plane: $0 = A(x-x_0) + B(y-y_0) + C(z-z_0)$.

Def^n: Given a function $z = f(x,y)$, the differential of $f(x,y)$, written dz, is given by:

$$dz = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$$

Think of $dx \approx \Delta x = x - x_0$

$dy \approx \Delta y = y - y_0$.

$$dz \approx \Delta z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

This is called the Tangent Plane Approximation of $f(x,y)$ at the point (x_0,y_0,z_0).

Ex. Let $f(x,y) = x^2 - 2y^2 + \tan(xy) + x^{-1}$

(i) Find the diff^l of $f(x,y)$
(ii) Write the T.P.A. of $f(x,y)$ at $(x_0,y_0) = (-2,0)$
(iii) Estimate $f(-1.95, -0.02)$.

(i) $f_x(x,y) = 2x + \sec^2(xy)\cdot y - x^{-2}$
$f_y(x,y) = -4y + \sec^2(xy)\cdot x$

$$dz = (2x + \sec^2(xy)\cdot y - x^{-2})\,dx + (-4y + \sec^2(xy)\,x)\,dy$$

(ii) $f_x(x_0,y_0) = f_x(-2,0) = -4.25$
$f_y(x_0,y_0) = f_y(-2,0) = -2$.

$\Delta z = z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$
$z - z_0 = -4.25(x-(-2)) + (-2)(y)$
$z - z_0 = -4.25(x+2) - 2y$.

The T.P.A. @ $(x_0,y_0) = (-2,0)$

where $z_0 = f(x_0,y_0) = 3.75$.

(iii) Estimate $f(-1.95,-0.02)$. So, in our T.P.A., set $(x,y) = (-1.95,-0.02)$

$f(-1.95,-0.02) \approx z$
$= z_0 - 4.25(x+2) - 2y$
$= 3.75 - 4.25(-1.95+2) - 2(-0.02)$
$\approx 3.75 + (-0.1725)$
≈ 3.5775.

should be okay as an approximation since $\Delta x = 0.05$, $\Delta y = -0.02$

The Chain Rule: Recall: $f(x)$, where $x = x(t)$, where t is some other variable.

What is $\frac{df}{dt}$?

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt}$$

A similar idea applies to higher dimensions!

Ex. Let $z = f(x,y) = xy + \sin(x)$, where $x = u+v$, $y = u-v$.

Can we find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$?

Well, $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du}$

$= (y + \cos(x))(1) + (x)(1)$

$= x + y + \cos(x)$.

Usually, it helps to draw the following diagram:

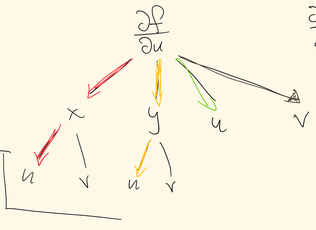
$\frac{\partial z}{\partial u}$

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$= (y + \cos(x))(1) + (x)(-1)$

$= -x + y + \cos(x)$.



Ex. Let $z - \arctan(z) = x + y$.

We cannot write $z = (x,y)$

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

So, take $\frac{\partial}{\partial x}$ on both sides!

$$\frac{\partial}{\partial x}\left(z - \arctan(z)\right) = \frac{\partial}{\partial x}(x+y)$$
$$= 1 + 0$$

$\frac{\partial}{\partial x}(f) = \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$

$\frac{\partial z}{\partial x} - \frac{\partial}{\partial x}(\arctan(z)) = \frac{\partial z}{\partial x} - \frac{d}{dz}(\arctan(z))\cdot\frac{\partial z}{\partial x}$

$= \frac{\partial z}{\partial x} - \left(\frac{1}{1+z^2}\right)\cdot\frac{\partial z}{\partial x}$.

$\frac{\partial z}{\partial x} - \left(\frac{1}{1+z^2}\right)\frac{\partial z}{\partial x} = 1$

$\frac{\partial z}{\partial x}\left[1 - \frac{1}{1+z^2}\right] = 1$

$\frac{\partial z}{\partial x}\left[\frac{z^2}{1+z^2}\right] = 1$

$\frac{\partial z}{\partial x} = \frac{1+z^2}{z^2} = 1 + z^{-2}$.

$\frac{\partial z}{\partial y} \longrightarrow \frac{\partial z}{\partial y} - \frac{1}{1+z^2}\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x+y)$

$= 1$

$\frac{\partial z}{\partial y} = 1 + z^{-2}$.