Lab 2

Thursday, May 7, 2020 3:48 PM

Topics:
- Quick review of domain, range, graphs & level curves
- limits & continuity
- partial derivatives

Recall: Given a function $z = f(x_1, \dots, x_n)$, the domain of f is all possible allowable inputs $(x_1, \dots, x_n)$, and the range of f is all possible outputs given $(x_1, \dots, x_n) \in \text{Dom}(f)$.

The graph of a function is the set of all points $(x_1, \dots, x_n, z) \in \mathbb{R}^{n+1}$ such that $f(x_1, \dots, x_n) = z$ for some element in the domain of f.

Level curves of a function $f(x_1, \dots, x_n)$ are all the points in the domain such that $f(x_1, \dots, x_n) = K$ for a fixed constant K.

Ex: Sketch the following and describe the level curves:

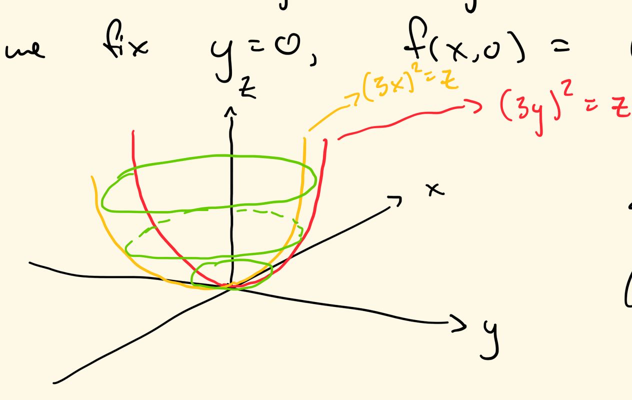
$$f(x,y) = 4x^2 + 9y^2 = (2x)^2 + (3y)^2.$$

First, notice that $f(x,y) \geq 0$. If we fix $x = 0$,

$$f(0,y) = (3y)^2 \quad \text{(a parabola).}$$

If we fix $y = 0$, $f(x,0) = (2x)^2$ (also a parabola).

So,



So this is a paraboloid, centred at $(0,0)$, opening upwards.

In green, we see the level sets are ellipses, which are getting larger as the constant K increases.
Mathematically, for fixed K, the curve is $K = (2x)^2 + (3y)^2$, which is indeed an ellipse!

Limits: Remember that in higher dimensions, the limit must be the same from every direction! Given a question that says "Does the limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exist?", the general strategy is:

1. Show that the limit is different from different directions, which means the limit D.N.E.
   Ex. Set $x = 0$, take the limit, set $y = 0$, take the limit, show they are different.
   } limit D.N.E.

or:

2. Show that the limit is the same from all directions!
   Ex. Set $y = mx$ for any $m \in \mathbb{R}$. Take the limit as $x \to a$ and show that the limit does NOT depend on the choice of m.
   } limit exists, probably.

Unfortunately, even in the second case we still need to do more to show that the limit exists! One method is to use polar coordinates.

Ex. Find $\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+3y^2}$.

First, try $y = mx$: $\lim_{(x,y)\to(0,0)} \frac{x^2 m^3 x^3}{2x^2 + 3m^2x^2} = \lim_{(x,y)\to(0,0)} \frac{m^3 x^5}{x^2(2+3m^2)} \longrightarrow 0.$

So, it looks like the limit is zero. Set $x = r\cos\theta$, $y = r\sin\theta$.

Then,
$$f(r,\theta) = \frac{r^2\cos^2\theta \, r^3 \sin^3\theta}{2r^2\cos^2\theta + 3r^2\sin^2\theta} = \frac{r^3\cos^2\theta\sin^3\theta}{2\cos^2\theta + 3\sin^2\theta}.$$

Then, we simplify the denominator:
$$f(r,\theta) = \frac{r^3\cos^2\theta\sin^3\theta}{2(\cos^2\theta + \sin^2\theta) + \sin^2\theta} = \frac{r^3\cos^2\theta\sin^3\theta}{2 + \sin^2\theta}.$$

Now we squeeze $f(r,\theta)$. We will use two facts:

$$-1 \leq \cos^2\theta\sin^3\theta \leq 1, \quad \text{and}$$
$$2 \leq 2 + \sin^2\theta \leq 3.$$

Thus,
$$-\frac{r^3}{3} \leq f(r,\theta) \leq \frac{r^3}{2}$$

Taking $\lim_{r\to 0} f(r,\theta) = 0$ proves the limit is indeed zero!

Recall: If a function has a limit at a point $(x_1, \dots, x_n)$, the function is said to be CONTINUOUS at $(x_1, \dots, x_n)$.
If a function has a limit at every point in some set $D \subset \mathbb{R}^n$, the function is said to be CONTINUOUS on D.

Partial Derivatives: When we are given a function $f(x_1, \dots, x_n)$ of two or more variables, we can still talk about derivatives, but now it depends on what variable you want to take the derivative with respect to.

In general, the partial derivative of a function $f(x_1, \dots, x_n)$ with respect to the x_i^{th} at $x_i = a$ is

$$\lim_{h\to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

This is the same as in one dimension, but now we treat all remaining variables as constant! Let's do examples to see.

Ex. Find $\frac{\partial f}{\partial y}$ of $f(x,y) = x^2 + y^2$.

As said previously, treat the remaining variables as constant. In this case, x is treated as a constant as we want the partial derivative with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = 0 + 2y = 2y.$$

Ex. Find $\frac{\partial f}{\partial y}$ of $f(x,y) = x^2 + xy + y^2$.

Well,
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + xy + y^2) = 0 + x + 2y = x + 2y.$$

Ex. Find $\frac{\partial f}{\partial x}$ of $x^2e^{xy} + \sin(ax)\cdot y$

Well,
$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2e^{xy} + \sin(ax)y\right), \text{ treat } y \text{ as constant:}$$
$$= \frac{\partial}{\partial x}(x^2e^{xy}) + y\cdot\frac{\partial}{\partial x}(\sin(ax))$$
$$= 2xe^{xy} + x^2ye^{xy} + ay\cos(ax).$$

Ex. Show that $u(x,y) = x^2\sin(y) + y^2\sin(x)$ satisfies
$$u_x + u_{xy} = 2x(\sin(x) + \sin(y)) + y(y+2)\cos(x).$$

Well, $\frac{\partial u}{\partial x} = 2x\sin(y) + y^2\cos(x)$. Then,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = u_{xy} = 2x\cos(y) + 2y\cos(x)$$

Add and simplify to see that the equation is satisfied!