Lab 1

Saturday, May 2, 2020 12:38 PM

<u>Topics</u>:
- Domain/range of a function
- level curves / graphs
- limits, continuity and ~~partial derivatives~~ → next lab!

Some informal definitions:

The <u>domain</u> of a function are <u>all</u> values that you can <u>legally</u> input into the function.

Ex. $f(x) = \sqrt{x}$. If $x = 1$, $f(x) = 1$
$x = 2$, $f(x) = \sqrt{2}$.

Both outputs are legal for $x = 1, x = 2$.

If $x = -1$, $f(-1) = \sqrt{-1}$ is not defined (in this course).

So, -1 is not a legal input.

This means that $x = 1, 2$ are in the domain of $f(\cdot)$,

$x = -1$ is <u>not</u> in the domain of $f(\cdot)$.

<u>Question</u>: What is the domain of $f(x) = \sqrt{x}$?

The <u>range</u> of a function $f(x)$ are all possible values that the function $f(x)$ returns as an output.

Ex. Is 1 in the range of $f(x) = \sqrt{x}$? Yes! Since $f(1) = 1$.

Is -1 in the range of $f(x)$? No! $f(x) \geq 0$ for our domain.

<u>Q</u>: What is the range of $f(x)$?

Domain/Range review: find the domain/range of each function.

1. $f(x) = x^2$
2. $f(x) = \ln(x)$
3. $f(x) = \sin(x)$
4. $f(x) = \tan(x)$

The same idea applies to a <u>multi variable</u> function! That is, given a function $f(x,y)$ which now takes <u>two</u> inputs (x,y) and returns a third value z, the <u>domain</u> is all values (x,y) that $f(x,y)$ can legally take as inputs, and the range is all values that can be reached by $f(x,y)$.

Ex. $f(x,y) = x\ln(y^2 - x)$. Find the domain of $f(x,y)$.
Let's tackle this piece by piece. x out front can be any real number, but the value inside the log must be positive, since the natural log of a negative number is not defined (in this course). So,

$$y^2 - x > 0.$$ Thus,

The domain of $f(x)$, $\text{Dom}(f)$, $= \{(x,y) : y^2 - x > 0\}$.

Ex. Find the domain & range of $g(x,y) = \sqrt{9 - x^2 - y^2}$

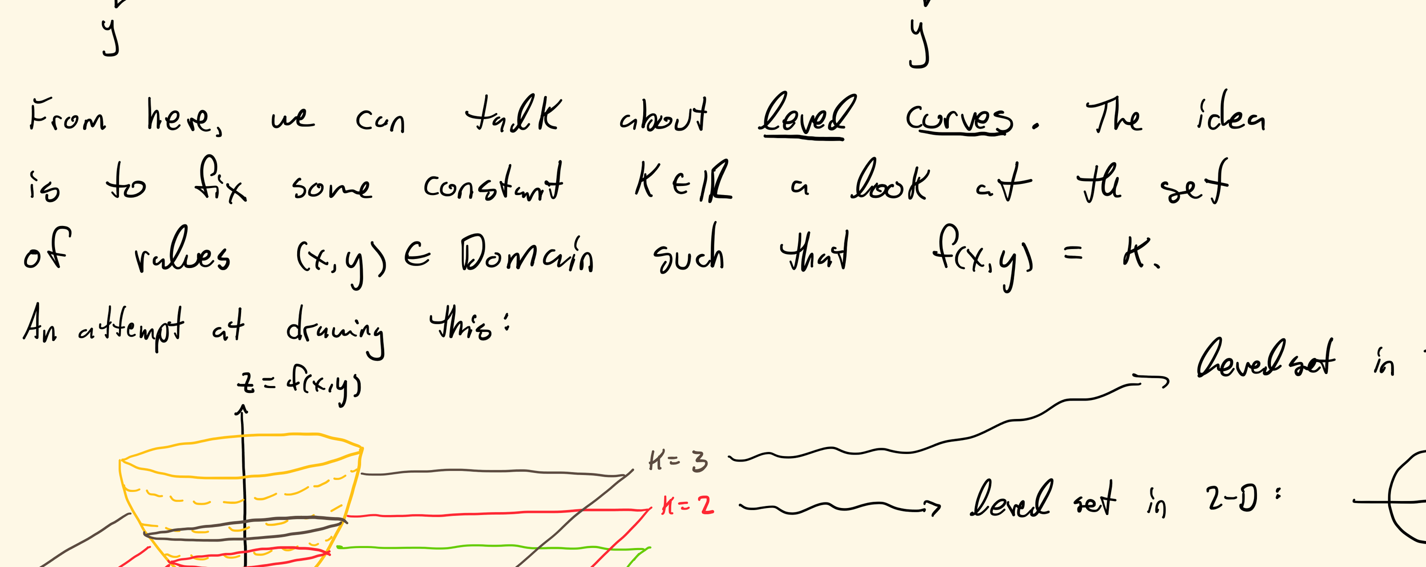
Since the quantity under the square root cannot be negative,
$\text{Dom}(f) = \{(x,y) : x^2 + y^2 \leq 9\}$.

Notice that $g(x,y) \geq 0$ due to the domain above, and choosing $x = y = 0$, $f(0,0) = \sqrt{9} = 3$, the largest possible value $g(x,y)$ can reach. Hence, $0 \leq g(x,y) \leq 3$, and the range of g, $\text{Range}(g) = \{z : 0 \leq z \leq 3\}$
$= [0,3]$.

Now that we have domain and range understood, we can talk about the <u>graph</u> of a function $f(x,y)$.

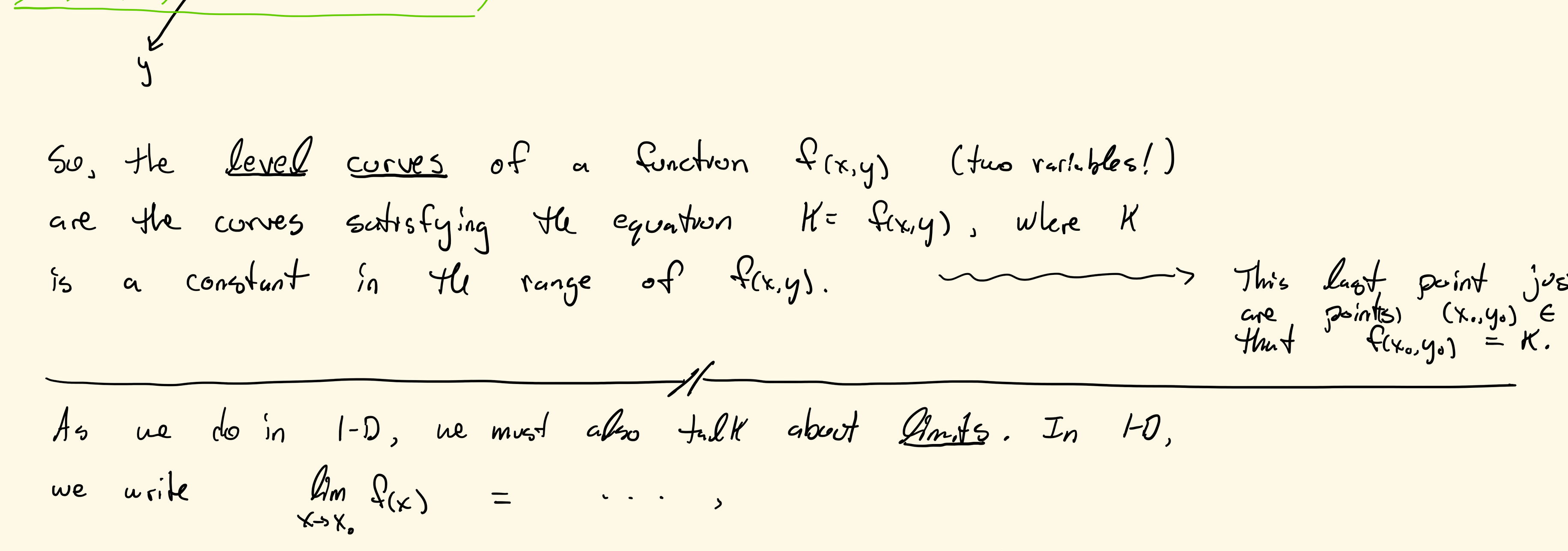
The <u>GRAPH</u> of a function $f(x,y) = z$ is the set of all points $(x,y,z) \in \mathbb{R}^3$ such that $z = f(x,y)$ and $(x,y) \in \text{Dom}(f)$.

In 2-D, $y = f(x)$ and the graph is just a curve. In 3-D, the graph will become a <u>surface</u>!



From here, we can talk about <u>level curves</u>. The idea is to fix some constant $K \in \mathbb{R}$ a look at the set of values $(x,y) \in$ Domain such that $f(x,y) = K$.

An attempt at drawing this:



So, the <u>level curves</u> of a function $f(x,y)$ (two variables!) are the curves satisfying the equation $K = f(x,y)$, where K is a constant in the range of $f(x,y)$. → This last point just means there are point(s) $(x_0,y_0) \in \text{Dom}(f)$ such that $f(x_0,y_0) = K$.

As we do in 1-D, we must also talk about <u>limits</u>. In 1-D, we write

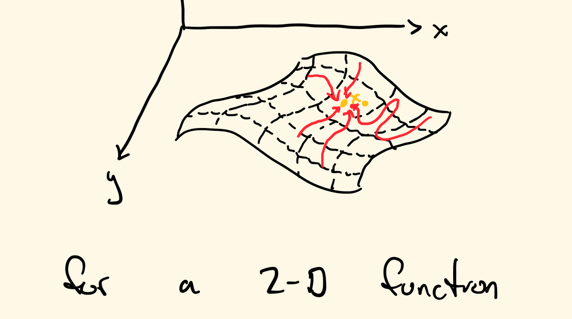
$$\lim_{x \to x_0} f(x) = \ldots,$$

and we may have learned that this limit exists if

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x).$$

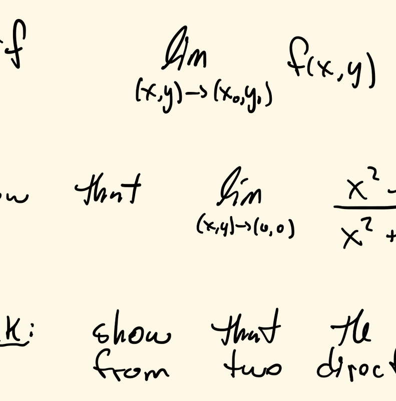
This is good, but in 2-D there are <u>many</u> directions you can approach a point from! Consider a picture:

<u>1-D</u>:



<u>two</u> directions

<u>2-D</u>:



<u>many</u> directions!

So, for a 2-D function $f(x,y)$, the limit exists only if $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ is the <u>same</u> from <u>every direction</u>!

<u>Ex.</u> Show that $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ D.N.E.

<u>The trick</u>: show that the limit is different from two directions.

So, set $y = 0$ and approach $x \to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = 1$$

Set $x = 0$ and approach $y \to 0$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = -1$$

$1 \neq -1$, so this limit does not exist!

Ex. Does $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$ exist?

<u>The trick</u>: Set $y = mx$ and show the limit is the same for <u>any</u> m (and therefore the limit is the same from every direction).

$$\longrightarrow \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{(x,y)\to(0,0)} \frac{m^2 x^3}{x^2 + (mx)^4}$$

$$= \lim_{x \to 0} \frac{m^2 x^3}{x^2(1 + m^4 x^2)}$$

$$= \lim_{x \to 0} \frac{m^2 x}{1 + m^4 x^2} = 0$$

So, the limit exists and it is zero!

Now that we have limits in 2-D, we can talk about <u>continuity</u>.

In 1-D, if $\lim_{x \to x_0} f(x)$ exists for $x_0 \in \text{Dom}(f)$, f is said to be <u>continuous</u> at x_0.

We then say $f(x)$ is continuous on (a,b) if f is continuous for every point $x_0 \in (a,b)$.

In 2-D, a function is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

Then, $f(x,y)$ is <u>continuous on $D \subset \mathbb{R}^2$</u> if $f(x,y)$ is continuous at every point $(x,y) \in D$.