Bonus Lab II

Tuesday, December 3, 2019 3:02 PM

Topic: - Wave Equation

Ex. Solve:

$$\begin{cases} u_{tt} = 16 u_{xx}, & 0 < x < \pi, \ t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = \sin^2(x), & 0 < x < \pi \\ u_t(x,0) = 1 - \cos(x), & 0 < x < \pi \end{cases}$$

As usual, assume $u(x,t) = X(x)T(t)$.

If we plug this in, we have

$$T''X = 16TX''$$

or

$$\frac{T''}{16T} = \frac{X''}{X} = -\lambda$$

So, we now solve:

$$\begin{cases} T'' = -16\lambda T \\ X'' = -\lambda X \end{cases}$$

Start with the equation for X. If $\lambda = 0$, $X'' = 0$

$$\Rightarrow X = Ax + B.$$

Then, our boundary conditions imply $A = B = 0$.

If $\lambda < 0$, $X(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$,

and boundary conditions $\Rightarrow c_1 = c_2 = 0$.

Now, when $\lambda > 0$, $r = \pm i\sqrt{\lambda}$.

Then, $X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$.

Then, $X(0) = c_1 = 0$.

$X(\pi) = c_2\sin(\sqrt{\lambda}\pi)$

$\Rightarrow \sqrt{\lambda}\pi = n\pi$, or

$\lambda_n = n^2$.

Hence, $X_n(x) = c_n\sin(nx)$.

Now, return to the equation for T.

We have that

$$T_n'' = -16n^2 T_n$$

So, $r = \pm i4n$, and

$$T_n(t) = a_n\cos(4nt) + b_n\sin(4nt)$$

Hence, our solution is:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} X_n(x)T_n(t) \\ &= \sum_{n=1}^{\infty} c_n\sin(nx)\left[a_n\cos(4nt) + b_n\sin(4nt)\right] \end{aligned}$$

We now solve for the coefficients using the initial conditions.

Assume $c_n = 1$ (since c_n distributes to a_n and b_n, we are just relabelling coefficients).

Then,

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} \sin(nx)\, a_n \\ &= \sin^2(x). \end{aligned}$$

So, we want the sin series of $\sin^2(x)$ on the interval $(0,\pi)$. So,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} \sin^2(x)\sin(nx) \\ &= \frac{2}{\pi}\left[\frac{2((-1)^n - 1)}{n(n^2-4)}\right] \\ &= \frac{2}{\pi}\cdot\begin{cases} 0, & n \text{ even (do } n=2 \text{ case separately)}. \\ \dfrac{-4}{n(n^2-4)}, & n \text{ odd}. \end{cases} \end{aligned}$$

(This is not a fun time to integrate. I won't write it here as it is longer than the problem itself so far).

Then,

$$\begin{aligned} \frac{\partial u}{\partial t}(x,0) &= \sum_{n=1}^{\infty} \sin(nx)\cdot b_n \\ &= 1 - \cos(x). \end{aligned}$$

So, we now want the sin series of $1 - \cos(x)$ on $(0,\pi)$.

So,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi}(1-\cos(x))\sin(nx)\,dx \\ &= \frac{2}{\pi}\left(-\frac{\cos(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi}\cos(x)\sin(nx)\,dx\right) \\ &= \frac{2}{\pi}\left(\frac{1-(-1)^n}{n} - \left(\frac{n(\cos(nx)+1)}{n^2-1}\Big|_0^{\pi}\right)\right) \\ &= \frac{2}{\pi}\left(\frac{1-(-1)^n}{n} - \left(\frac{n((-1)^n+1)}{n^2-1} - \frac{2n}{n^2-1}\right)\right) \\ &= \frac{2}{\pi}\left(\frac{1-(-1)^n}{n} + \frac{2n}{n^2-1} - \frac{n((-1)^n+1)}{n^2-1}\right) \end{aligned}$$

So, our final solution is:

$$u(x,t) = \sum_{n=1}^{\infty}\sin(nx)\left[a_n\cos(4nt) + b_n\sin(4nt)\right],$$

with the coefficients found above.